Algebraic Quantum Statistical Mechanics and Bose–Einstein Condensation

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Tiivistelmä — Referat — Abstract

This thesis can be regarded as a light, but thorough, introduction to the algebraic approach to quantum statistical mechanics and a subsequent test of this framework in the form of an application to Bose-Einstein condensates.

The success of the algebraic approach to quantum statistical mechanics hinges upon the remarkable properties of special operator algebras known as C^*-algebras. These algebras have unique characterization properties which allows one to readily identify the mathematical counterparts of concepts in physics while at the same time maintaining mathematical rigour and clarity.

In the first half of this thesis, we focus on abstract C^*-algebras known as the canonical commutation relation algebras (CCR algebras) which are generated by elements satisfying specific commutation relations. The main result in this section is the proof of a certain kind of algebraic uniqueness of these algebras. The main idea of the proof is to utilise the underlying common structure of any of the CCR algebras and explicitly construct an isomorphism between the generators of these algebras. The construction of this isomorphism involves the use of abstract Fourier analysis on groups and various arguments concerning bounded operators.

The second half of the thesis concerns the rigorous set-up of the formation of Bose-Einstein condensation. First, one defines the Gibbs grand canonical equilibrium states, and then we specialize to studying the taking of the thermodynamic limit of these systems in various contexts. The main result of this section involves two main elements. The first is that by fixing the temperature and density of the system while varying its activity and volume, there exists a limiting state corresponding to the taking of the thermodynamic limit. The second element concerns the existence of a critical density after which the limiting state begins to show the physical characteristics of Bose-Einstein condensation.

The mathematical issues one faces with Bose-Einstein condensation are mainly related to the unboundedness of the creation and annihilation operators and the definition of the algebra that we are working on. The first issue is relevant to all areas of mathematical physics, and one deals with it in the standard ways. The second issue is more nuanced and is a direct result of the first issue we mentioned. In particular, we would like to define the states on an algebra which contains the operators that we are interested in. The problem is that these operators are unbounded, and, as a result, one must instead use the CCR algebra and show by extension that we can, in fact, also use the unbounded operators in this state.

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Contents

1 Introduction

This thesis is a light, but thorough, introduction to the algebraic approach to quantum statistical mechanics and a subsequent application to a non-trivial, but relevant, physical system.

Statistical mechanics, both quantum and classical, aims to clarify the structure of the macroscopic theory of thermodynamics. The relationship between thermodynamics and statistical mechanics is akin to the relationship between Newtonian mechanics and Hamiltonian mechanics. In some sense, Hamiltonian mechanics is a generalization or a clarification of Newtonian mechanics which has a considerable richer structure which to study. In the same way, although statistical mechanics gives us answers to similar questions as thermodynamics, the theory of statistical mechanics itself is considerably richer and allows one to describe additional structure which is not present in thermodynamics.

In the context of this thesis, we will be interested in equilibrium quantum statistical mechanics. The foundations of statistical mechanics require a considerable amount of mathematical theory and applications thereof. In particular, to rigorously proceed beyond very elementary theorems, one must justify the ergodic hypothesis. Unfortunately, this is usually highly non-trivial and is a completely different topic to the brunt of this thesis.

To one not well-versed in statistical mechanics, many of the methods and definitions in this thesis will seem arcane. In a sense, the algebraic formalism is a framework in which we can rigorously do statistical mechanics. The downside of this approach is that it is very abstract, and, secondly, concrete realizations of some of these concepts are very difficult to construct.

There is also a flavour of arbitrariness to some of the definitions that will be provided. This flavour occurs because of the high-level approach to some problems. By high-level, we mean that there is no necessary "foundational" reason to specifically study the given object. This will occur with the so-called Gibbs grand canonical equilibrium state. For this state, we will simply specify an equation which will define the state, extend this state to some relevant objects so that we can do some computations, and utilize this state to define other states. In all of these cases, there is a distinct lack of uniqueness in these definitions and the subsequent states which makes the theorems feel somehow empty.

Of course, the theorems themselves are highly non-trivial, and one can consider it an achievement to be able to even define a relevant structure in which one can deduce real physical consequences.

The motivated reader is urged to peruse [8, Part 1, Chapters 1-5] for a sufficient background to understand the statistical mechanics required for this thesis. In particular, one can focus on understanding the fundamental and mathematical differences between thermodynamics and statistical mechanics.

The main mathematical theories that will be utilized and explored in this thesis are functional analysis and C^*-algebras.

One can view C^*-algebras as "sufficiently regular" algebras of operators. The reason for this regularity is because of the topological properties of such algebras. For instance, in regular functional analysis one makes a distinction between strong convergence and weak convergence. In some situations, weak convergence is a sufficient and necessary condition for some theorems. In fact, the inclusion of strong continuity might even trivialize the theorem to the point that there are only trivial examples of the theorem. In the same sense, C^*-algebras are sufficient, but not restrictive, for theory building.

Historically, von Neumann and Murray were the first mathematicians to study operator algebras on Hilbert spaces in the 1930's. However, they studied what became to be known as von Neumann algebras or W^*-algebras which are *-algebras that are weak-star closed. It was not until 1943 due to the investigations of Gelfand and Naimark that C^*-algebras were defined and studied in detail. Initially, in 1932, von Neumann suggested the use of so-called Jordan algebras for algebras of observables. At the time, Jordan algebras lacked characterization theorems, and, as a result, in 1947, C^*-algebras became a standard tool in the theory of operator algebras due to work on various characterizations and correspondence theorems due to Gelfand, Naimark, and Segal.

The previous paragraph is a short synthesis of [1, pp. 1-7]. For a more thorough understanding of the history of operator algebras and the development of quantum mechanics along with quantum statistical mechanics, we suggest [1, pp. 1-15].

One of the considerable benefits of working in abstract C^*- and Banach-algebras is that one has a large host of classification theorems at hand. As an additional bonus, many of these classification theorems illuminate certain features of theory of bounded operators of Hilbert spaces. Abelian C^*- algebras with units are isomorphic to the space of continuous functions on a compact Haussdorf set [4, p. 236]. C^*-algebras are isomorphic to a closed subalgebra of some bounded operators on a Hilbert space [1, p. 24]. Using these two classification theorems, one can construct a useful precursor to the general spectral theory of normal operators.

In fact, the intuition for why normal operators are in a sense the "weakest" operators that we can construct a spectral theory for comes precisely from the theory of operator algebras. The reason is that in order to give a suitable interpretation for a Banach space valued function, one must somehow invoke the representation theorems for Abelian C^*-algebras. The algebra which is generated by an element and its adjoint is precisely an Abelian C^*-algebra which is the starting point for the spectral theory of normal operators.

The first half of this thesis will be concerned with the algebraic framework of specific C^*-algebras. The latter half will be concerned with a specific application to a non-trivial quantum statistical system.

We will study and prove numerous features of so-called Bose-Einstein condensates. These condensates form when certain conditions regarding physical parameters of a system are met. In particular, a certain density, temperature, and activity of the system must be maintained to give a sufficient description of this phenomenon. The main purpose of the latter half is to be able to show the existence of a change of phase from a regular gas with properties that you would expect from a regular gas to the condensate phase in which certain bizarre quantum effects become pertinent.

Perhaps the most striking feature of the condensate phase is that there is a certain scale invariance between the correlations of particles. This will be seen by studying the two-point correlations between particles, and checking what happens when two particles are created arbitrarily far away from each other. For a regular gas, we would expect the correlations to vanish. Physically, this would correspond to there being no "interaction" between the particles. However, for the condensate, regardless of the distance between the created particles, there will always be a non-zero correlation.

Throughout this thesis, there will be elements and calculations of physics, but, for the most part, we will aim to be completely rigorous and prove, or at least sketch the proof, to all relevant statements. When it is necessary, we will give references to sufficiently high-level results, and, so as to not spend too much time on tedious but trivial notions, we will occasionally give references for simpler statements.

One can consider this thesis to be a clarification of the first section of [1]. If the reader is familiar with this work, then they will recall the sparseness of detail present in this work. All the relevant information is given, but the details of most computations and proofs are left to the reader. Perhaps one of the most important clarifications made in this thesis is to the section concerning the bosonic Gibbs state. There are also certain marked differences in some of the proofs, mostly in the form of simplifications with easier assumptions so as to not obscure the main theorem.

The brunt of this thesis comes from [1], however, during the course of this work, I also found [11] to be an essential source. The seminar notes by Dénes Petz have more exposition into the algebraic approach to some proofs. However, both [11] and [1] use some "standard" techniques which can seem quite arcane to a novice of C^*-algebras.

We will briefly outline the more specific structure of this thesis, and give a reading guide.

We begin with section 2 by introducing the main areas of mathematics that will be used. These are functional analysis, abstract Harmonic analysis, and C^*-algebras. This section does not contain a complete list of all the results used, instead, it can be regarded as a synthesis of the most important, and possibly less well known, results used in this thesis.

In section 3, we construct the primary ambient Hilbert space, known as the symmetric Fock space or Bosonic Fock space, and subsequently we define the Weyl operators on this space. This constructed space will then later on be used for Bose-Einstein condensation.

The main algebraic work of this thesis is done in section 4. In this section, we deal with the abstract Weyl operators and give proofs for existence and algebraic uniqueness of the generated C^*-algebras.

We continue with the algebraic part of quantum statistical mechanics by specializing to more regular states in section 5. In particular, we specify some analyticity and continuity properties of the representations of the time evolutions of the states to construct abstract annihilation and creation operators.

Finally, in section 6.1, we study the Gibbs grand canonical equilibrium states and work through various mathematical technicalities to define the desired state. Equipped with these finite volume Gibbs states, we then study the taking of the thermodynamic limit. This is done in multiple context with different fixed variables. In the end, the main result of this section is that we show that by having a fixed density and temperature, while varying the activity of the system, we are able to give a satisfactory description of the limiting state, and the desired qualities of Bose-Einstein condensation are present.

In section 7, we give a summary of the main results, and suggest some further works to study and explore for the motivated readers.

2 Preliminaries

2.1 Functional Analysis

The necessary functional analysis to understand this thesis can mostly be found in [12] and [15].

One of the most important tools is spectral theory. Spectral theory in itself is an extremely broad and complex set of definitions and theorems which concerns the development of functional calculus for normal operators. The motivated reader is suggested to go through [15, pp. 306-375] for a complete understanding of the topic. We will present the main result here.

Theorem 2.1. *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. There exists a unique* *resolution E of the identity, on the Borel subsets of the real line, such that*

$$\langle \phi, T\psi \rangle = \int_{\mathbb{R}} dE_{\phi,\psi}(\lambda)\ \lambda\ . \tag{2.1}$$

Futhermore, if $f : \mathbb{R} \to \mathbb{C}$ is measurable, then

$$\langle \phi, f(T)\psi \rangle := \int_{\mathbb{R}} dE_{\phi,\psi}(\lambda)\ f(\lambda)\ . \tag{2.2}$$

The above theorem is known as the spectral theorem, and the second form is known as functional calculus.

The readier is urged to read [3][pp. 397-404] to gain a sufficient understanding of Bochner integrals. Our main intent is to have a satisfactory theory of the integration of Banach valued integrals. We will use the dominated convergence theorem for Bochner integrals.

Theorem 2.2. *Let (X, σ, μ) be a measure space, let E be a real or complex Banach space and let $g : X \to [0, \infty]$ be an integrable function. Suppose that f and $\{f_n\}_{n\in\mathbb{N}}$ are strongly measurable E-valued functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{2.3}$$

and $||f_n(x)|| \leq g(x)$ for almost all x.

It follows that f and $\{f_n\}_{n\in\mathbb{N}}$ are integrable, and

$$\lim_{n\to\infty} \int d\mu\ f_n = \int d\mu\ f\ . \tag{2.4}$$

Finally, we will need some specific theorems concerning Hilbert-Schmidt operators.

Definition 2.1. *Let T be a bounded operator on a Hilbert space $\mathcal{H}$. We say that T is a Hilbert-Schmidt operator if $\mathrm{Tr}(T^*T) < \infty$.*

Next, we will list some of the important properties of such operators.

Theorem 2.3. *Let A be a Hilbert-Schmidt operator and B a bounded operator. We have*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)\ . \tag{2.5}$$

The operator A is compact and if A is self-adjoint, then

$$\lim_{n\to\infty} \lambda_n^2 \to 0 \tag{2.6}$$

for the eigenvalues of A.

2.2 C^*-algebras

Our aim here is to collect a number of results that will be needed and to give a general overview of the theory of C^*-algebras as it pertains to this thesis. For the interested and motivated reader is suggested to read the section on Banach algebras [13] for a very light introduction to Banach algebras which does not need any knowledge of general topology. Going further, a next step would be to read the chapters which pertain to C^*-algebras in [4]. This book, however, will require a good background in general topology, topological groups, and general functional analysis. Finally, for a more abstract and purely mathematical treatment of C^*-algebras and W^*-algebras, we recommend [10].

We encourage the reader to get a basic understanding of general Banach algebras. Here, we will only recite some of the more important results in C^*-algebras.

Definition 2.2. *Let $\mathcal{A}$ be a Banach algebra equipped with an involution mapping $a \mapsto a^*$. If $\mathcal{A}$ satisfies $||a^*a|| = ||a||^2$, then we call $\mathcal{A}$ a C^*-algebra.*

The reason that this area of mathematics is called operator algebras is because of the following characterization theorem.

Theorem 2.4. *Every C^*-algebra is isomorphic to a closed subalgebra of bounded operators of some Hilbert space.*

There are many interesting and relevant proofs for bounded operators on Hilbert spaces. The theory of C^*-algebras clarifies which of these proofs are, in fact, algebraic in their nature rather than analytic in the form of Hilbert spaces.

In this thesis, we will be considering objects defined on C^*-algebras called states. There is a very interesting construction which concerns these states. The following theorem defines what a state is and defines the vector representation of the state.

Theorem 2.5. *Let $\mathcal{A}$ be a C^*-algebra. A positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is called a state on $\mathcal{A}$.*

For every state ω on $\mathcal{A}$, there exists a Hilbert space $\mathcal{H}_\omega$, a cyclic vector Ω_ω, and a representation π_ω such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \ .$$

The previous theorem is called the Gelfand-Neimark-Segal construction of a vector state.

The main theme regarding C^*-algebras and the relevant objects are that they are reminiscent of interesting objects for bounded operators on some Hilbert space.

2.3 Abstract Fourier Analysis

A general knowledge of abstract Fourier analysis is not required to fully understand this thesis. In particular, we will mostly be using the generalization of the Fourier transform to locally compact Abelian groups. The main difference between regular Fourier analysis and the Fourier analysis required in this thesis is that one needs to pay attention to the structure of the space of characters of the domain of integration. In particular, finite dimensional spaces are isomorphic to their dual, and, as a result, one generally does not need to study the characters of the space too closely.

We will give a collection of general results and definitions from [14].

First, we will start with the most basic object of interest: the locally compact Abelian group.

Definition 2.3. *A group $(G, \cdot)$ equipped with a topology τ is a topological group if the mappings*

$$a \mapsto a^{-1}, \ (a, b) \mapsto a \cdot b \tag{2.7}$$

are continuous. We note that topology of the initial space of the second mapping is the product topology.

Definition 2.4. *A topological group where the underlying group $(G, \cdot)$ is Abelian and the topology τ is locally compact is called a locally compact Abelian group or a LCA.*

When it is obvious, we will omit the mention of the topology τ and the group operation $\cdot$. In particular, for the group operation, we will use the canonical multiplication and summation symbols if it is relevant to the structure at hand. Namely, we will use the $+$-sign to signify that a similar structure to something akin to vector space additions is being used.

In order to generalize the Fourier transform, we will need to define the dual group.

Definition 2.5. *Let G be a locally compact Abelian group. A complex function $\gamma : G \to \mathbb{C}$ is called a character if*

$$\forall x \in G,\ |\gamma(x)| = 1,\ \forall x, y \in G,\ \gamma(x+y) = \gamma(x)\gamma(y)\ . \tag{2.8}$$

The set of all continuous characters equipped with the operation

$$(\gamma_1 + \gamma_2)(x) := \gamma_1(x)\gamma_2(x) \tag{2.9}$$

forms an Abelian group which will be denoted by Γ and is called the dual group of G.

Next, we will define the Fourier transform using the dual group.

Definition 2.6. *Let G be a locally compact Abelian group and Γ be its dual group. Let $f \in L^1(G)$ where the group G is equipped with the up-to-factor unique Haar measure.*

For such an f, we define a mapping $\widehat{f} : \Gamma \to \mathbb{C}$ by

$$\widehat{f}(\gamma) := \int_G dx\ f(x)\gamma(-x)\ . \tag{2.10}$$

The mapping $\widehat{f}$ is called the Fourier transform of f.

To finish this discussion, we will need to specify a topology on Γ.

Theorem 2.6. *Let G be a locally compact Abelian group and let Γ be its dual group. Define the collection of Fourier transforms to be $A(\Gamma)$. Explicitly, we have*

$$A(\Gamma) := \{\widehat{f} : f \in L^1(G)\}\ . \tag{2.11}$$

The weak topology induced by $A(\Gamma)$ on Γ makes Γ into a locally compact Abelian group.

From here on out, we will always use this topology for the dual group.

Next, we specify a theorem which will be used later on concerning the topologies of the group and its dual.

Theorem 2.7. *Let G be a locally compact Abelian group and Γ its dual group. If G is discrete, then Γ is compact, and if G is compact, then Γ is discrete.*

Finally, we present the Pontryagin duality.

Theorem 2.8. *Let G be a locally compact Abelian group and let Γ be its dual group. For $\gamma \in \Gamma$, we define the dual bracket by*

$$\langle x, \gamma \rangle := \gamma(x)\ . \tag{2.12}$$

Define $\widehat{\Gamma}$ to be the dual group of Γ. Define the mapping $\alpha : G \to \widehat{\Gamma}$ by

$$\langle \gamma, \alpha \rangle (x) := \langle x, \gamma \rangle\ \ . \tag{2.13}$$

The mapping α is a homeomorphism and an isomorphism. The Pontryagin duality then refers to this natural identification which preserves all relevant structure.

These definitions and their proofs can all be found in [14, pp. 1-30].

3 Weyl Operators and Second Quantization

3.1 Second Quantization

Without proof, we will present a collection of theorems and definitions which will let us use and define the second quantization method.

Typically, given a Hilbert space $\mathcal{H}$ and a self-adjoint operator H, we call $\mathcal{H}$ the single particle space and H the single particle Hamiltonian. We are interested in utilizing these single particle spaces and the single particle Hamiltonian to construct a space and an operator which contains all the necessary information and dynamics of a system which contains any finite amount of particles and has dynamics described by each particle interacting with only the Hamiltonian H. This method is known as second quantization.

We begin with the definition of Fock space which we denote by $\mathcal{F}$. Let $\mathcal{H}^n := \bigotimes_{i=1}^n \mathcal{H}$ refer to the n-fold tensor product of $\mathcal{H}$ with itself. The direct sum of all these finite particle spaces is called Fock space. We define

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{H}^n \ , \tag{3.1}$$

where $\mathcal{H}^0 = \mathbb{C}$.

Let S_n be the set of n-permutations and define the operator $P_n^{(+)}$ on $\mathcal{H}^n$ by

$$P_n^{(+)}(f_1 \otimes ... \otimes f_n) := \frac{1}{n!} \sum_{\pi \in S_n} f_{\pi(1)} \otimes ... \otimes f_{\pi(n)} \ . \tag{3.2}$$

This is not a full definition, but one can extend $P_n^{(+)}$ to the whole space by extension by continuity and one notes that $P_n^{(+)}$ is a bounded operator with norm 1.

We define the operator $P^{(+)}$ on $\mathcal{F}$ by $P^{(+)} := \bigoplus_{n=0}^{\infty} P_n^{(+)}$ and we define the symmetric Fock space which will be denoted $\mathcal{F}^{(+)}$ by

$$\mathcal{F}^{(+)} := P^{(+)}\mathcal{F} \ . \tag{3.3}$$

We can also define the symmetric n-particle spaces by $\mathcal{H}_n^{(+)} := P^{(+)}\mathcal{H}^n$ in which case we can alternatively write

$$\mathcal{F}^{(+)} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{(+)} \ . \tag{3.4}$$

Using the single particle Hamiltonian H, we define an operator H_n on $\mathcal{H}_n^{(+)}$ by

$$H_n(P^{(+)}(f_1 \otimes ... \otimes f_n)) := P^{(+)} \left(\sum_{i=1}^{n} f_1 \otimes ... \otimes H f_i \otimes ... \otimes f_n \right) \ . \tag{3.5}$$

Note that we must have $f_k \in D(H)$. Given such a definition on the tensor products, we define the second quantization of H to be

$$d\Gamma(H) := \overline{\bigoplus_{n=0}^{\infty} H_n} \ . \tag{3.6}$$

Without proof, we remark that the operator $d\Gamma(H)$ will be a self-adjoint operator with a dense domain defined on $\mathcal{F}^{(+)}$.

Given a unitary operator U on $\mathcal{H}$, one defines

$$U_n(P^{(+)}(f_1 \otimes ... \otimes f_n)) := P^{(+)}(Uf_1 \otimes ... \otimes Uf_n) \ . \tag{3.7}$$

We define the second quantization of U to be

$$\Gamma(U) := \bigoplus_{n=0}^{\infty} U_n \ . \tag{3.8}$$

The operator $\Gamma(U)$ is unitary and, furthermore, if $U_t = e^{itH}$ defined by the spectral theorem, then $\Gamma(U_t) = e^{itd\Gamma(H)}$.

The entire process of second quantization is the method by which we obtained the symmetric Fock space $\mathcal{F}^{(+)}$, the second quantized operator $d\Gamma(H)$ and the unitary operator $\Gamma(U)$. When dealing with symmetric permutations, we call these spaces Bosonic spaces. If one swaps the symmetric permutations for all anti-symmetric projections, we call that space the Fermionic space. However, for our purposes, we will only be interested in the Bosonic Fock space $\mathcal{F}^{(+)}$ and the second quantized operators on it.

Finally, we will define the symmetric annihilation and creation operators. First, we define the dense domain

$$D(N) := \{\oplus_{n=0}^{\infty}\psi_n : \sum_{n\in\mathbb{N}} n^2||\psi_n||^2 < \infty\} \subset \mathcal{F} \ . \tag{3.9}$$

This time, we need to be careful with the range of the operator. Let $f \in \mathcal{H}$. We define $a_n(f) : \mathcal{H}^n \to \mathcal{H}^{n-1}$ and $c_n(f) : \mathcal{H}^n \to \mathcal{H}^{n+1}$ by

$$a_n(f)(f_1 \otimes ... \otimes f_n) := n^{\frac{1}{2}} \langle f_1, f\rangle (f_2 \otimes ... \otimes f_n) \ , c_n(f) := (n+1)^{\frac{1}{2}}(f \otimes f_1 \otimes ... \otimes f_n) \ . \tag{3.10}$$

Now, one defines $a(f)$ and $c(f)$ on the dense domain $D(N^{\frac{1}{2}})$ by

$$a(f) := \bigoplus_{n=0}^{\infty} a_n(f), \ c(f) := \bigoplus_{n=0}^{\infty} c_n(f) \ . \tag{3.11}$$

Finally, we define

$$a_+(f) := P^{(+)}a(f)P^{(+)}, \ c_+(f) := P^{(+)}c(f)P^{(+)} \ . \tag{3.12}$$

Furthermore, one can show that $c_+(f) = a_+^*(f)$ and we have the following commutation relations

$$[a_+(f), a_+(g)] = 0 = [a_+^*(f), a_+^*(g)] \ , [a_+(f), a_+^*(g)] = \langle g, f\rangle \, \mathbb{1} \ . \tag{3.13}$$

As we stated earlier, all of these claims were made without proof. For some guidelines, one can refer to [1, pp. 6-13].

3.2 Weyl Operators

First, We will briefly introduce the field operators $\Phi(\cdot)$. The following lemma contains the necessary information

Lemma 3.1. *Let $f, g \in \mathcal{H}$, and define the bosonic creation and annihilation operators $a_+^*(\cdot)$ and $a_+(\cdot)$ in the standard way.*

Define the operator $\Phi(\cdot)$ on the finite particle vectors $F(\mathcal{H})$ by

$$\Phi(\cdot) = \frac{a_+(\cdot) + a_+^*(\cdot)}{\sqrt{2}} \ . \tag{3.14}$$

The following properties hold for the operator $\Phi(\cdot)$.

- *The operator $\Phi(\cdot)$ is essentially self-adjoint on the finite particle vectors $F(\mathcal{H})$, and, furthermore, the finite particle vectors form a dense set of analytic vectors of $\Phi(\cdot)$.*
- *Let $\Omega = (1, 0, ...) \in \mathcal{F}^{(+)}(\mathcal{H})$. The linear span of the set $\{\Phi(f_1)...\Phi(f_n)\Omega : f_1, ..., f_n \in \mathcal{H}\}$ is dense in $\mathcal{F}^{(+)}(\mathcal{H})$.*
- *For each $\psi \in D(N)$, where N is the standard number operator, we have*

$$(\Phi(f)\Phi(g) - \Phi(g)\Phi(f))\psi = i \operatorname{Im} \langle f, g\rangle \psi \ . \tag{3.15}$$

- *For $\psi \in \mathcal{H}_n^{(+)}$ and $m \in \mathbb{N}$, we have the following estimates*

$$\begin{aligned} &||a_+(f)\psi|| \leq (n+1)^{\frac{1}{2}}||f||||\psi|| \ , \\ &||a_+^*(f)\psi|| \leq (n+1)^{\frac{1}{2}}||f||||\psi||, \\ &||\Phi(f)^m\psi|| \leq 2^{\frac{m}{2}}(n+1)^{\frac{1}{2}}(n+2)^{\frac{1}{2}}...(n+m)^{\frac{1}{2}}||\psi||||f||^m \ . \end{aligned} \tag{3.16}$$

Proof. These operators and results are defined and proved in a standard course on quantum dynamics. The proofs and definitions can be found in [1, pp. 6-13]. □

From here on out, we will refer to $\Phi(\cdot)$ as the closure of the previously defined operator $\Phi(\cdot)$. That is, the previously defined operator is essentially self-adjoint on the finite particle vectors, and, as a result, its closure is a self-adjoint operator.

For any $f \in \mathcal{H}$, the operator $\Phi(f)$ is self-adjoint. Using the spectral representation, we can define a unitary operator $W(f)$ given by

$$W(f) := \exp(i\Phi(f)) \ . \tag{3.17}$$

The operators in the collection $\{W(f) : f \in \mathcal{H}\}$ are called the Weyl operators. The Weyl operators will be the primary operators of interest in the coming construction of the CCR algebra.

In this next proposition, we will prove some important properties of the Weyl operators.

Proposition 3.1. *For $\Phi(\cdot)$ and $W(\cdot)$ as defined previously, we have the following properties.*

1. *For any $f, g \in \mathcal{H}$, $W(f)D(\Phi(g)) = D(\Phi(g))$ and*

$$W(f)\Phi(g)W(f)^*\psi = \Phi(g)\psi - \operatorname{Im} \langle f, g\rangle \psi \ , \tag{3.18}$$

   for any $\psi \in D(\Phi(g))$.

2. *For any $f, g \in \mathcal{H}$, we have*

$$W(f)W(g)\psi = e^{-\frac{i \operatorname{Im}\langle f,g\rangle}{2}} W(f+g)\psi \tag{3.19}$$

   for any $\psi \in \mathcal{F}$.

3. *For any $f \in \mathcal{H}$ such that $f \neq 0$, we have*

$$||W(f) - \mathbb{1}|| = 2 \ . \tag{3.20}$$

Proof. We will prove the claims in the order they appear.

1. The idea of the proof will be to consider a suitable core on which we have a tractable representation of the operator $W(f)$, and then extend by linearity to the entire domain $D(\Phi(g))$.

Let $f, g \in \mathcal{H}$. By lemma 3.1, we know that the set of finite particle vectors $F(\mathcal{H})$ form an analytic dense subset of $D(\Phi(g))$. Let $\psi \in F(H)$. Since ψ is an analytic vector of $D(\Phi(g))$, we know that the spectral representation is an extension of the power series expansion of an operator on an analytic vector by [9, p. 410]. We have

$$W(f)\psi = \exp(i\Phi(f))\psi = \sum_{n=0}^{\infty} \frac{(i\Phi(f))^n}{n!}\psi = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!}\psi \ . \tag{3.21}$$

For $N \in \mathbb{N}$, we will need the following identity

$$\Phi(g)\frac{(i\Phi(f))^N}{N!}\psi = \frac{(i\Phi(f))^N}{N!}\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \frac{(i\Phi(f))^{N-1}}{(N-1)!}\psi \ . \tag{3.22}$$

We will prove eq. (3.22) by induction. For the base case $N = 1$, using the commutation relations for the operators $\Phi(\cdot)$ in lemma 3.1, we have

$$\Phi(g)i\Phi(f)\psi = i\Phi(f)\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \psi \ . \tag{3.23}$$

For the induction step, assume that the equality holds for some $N \in \mathbb{N}$. Using the same commutation relations as in the base case, we have

$$\begin{aligned}
\Phi(g)\frac{(i\Phi(f))^{N+1}}{(N+1)!}\psi &= \left(\Phi(g)\frac{(i\Phi(f))^N}{N!}\right)\frac{i\Phi(f)}{N+1}\psi \\
&= \frac{(i\Phi(f))^N}{N!}\Phi(g)\frac{i\Phi(f)}{N+1}\psi - \frac{\operatorname{Im}\langle f, g\rangle (i\Phi(f))^{N-1}}{(N-1)!}\frac{i\Phi(f)}{N+1}\psi \\
&= \frac{(i\Phi(f))^{N+1}}{(N+1)!}\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \frac{(i\Phi(f))^N}{(N+1)!}\psi - N\frac{\operatorname{Im}\langle f, g\rangle (i\Phi(f))^N}{(N+1)!}\psi \\
&= \frac{(i\Phi(f))^{N+1}}{(N+1)!}\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \frac{(i\Phi(f))^N}{N!}\psi \ .
\end{aligned} \tag{3.24}$$

Thus, the equality holds for $N + 1$, and hence for all natural numbers.

We remark that in the previous calculation, one need not pay too much attention to what elements the operators are acting on. In this case, $\Phi(\cdot)F(\mathcal{H}) \subset F(\mathcal{H})$, to be more precise, the induction proof should be done to include any vector $\psi \in F(\mathcal{H})$, and then the computations in the previous part are valid.

We are ready to compute the key identity. Applying eq. (3.22), we compute

$$\Phi(g)\sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!}\psi = \sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!}\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \sum_{n=0}^{N-1} \frac{(i\Phi(f))^n}{n!}\psi \ . \tag{3.25}$$

All that remains is to prove that the series that appear in eq. (3.25) converge to the desired operators for the finite particle vectors.

A previous computation in lemma 3.1 shows that

$$\sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!}\psi \in F(\mathcal{H}) \tag{3.26}$$

for all $N \in \mathbb{N}$, and

$$\sum_{n=0}^{\infty} \frac{||\Phi(g)\Phi(f)^n\psi||}{n!} < \infty \ . \tag{3.27}$$

We already know that the series in eq. (3.26) converges, and eq. (3.26) shows that the limit of the series satisfies $W(f)\psi \in D(\Phi(g))$ and the second property eq. (3.27) implies that the series

$$\Phi(g) \sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!} \tag{3.28}$$

converges. Using the previous two observations, and that $\Phi(g)$ is a closed operator, we must have

$$\lim_{N\to\infty} \Phi(g) \sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!} \psi = \Phi(g)W(f)\psi \ . \tag{3.29}$$

Next, since $\Phi(g)\psi \in F(\mathcal{H})$, and $\psi \in F(\mathcal{H})$ are analytic for $\Phi(f)$, we have

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{(i\Phi(f))^n}{n!} \Phi(g)\psi = W(f)\Phi(g)\psi \ . \tag{3.30}$$

Taking the limit in eq. (3.25), we have

$$\Phi(g)W(f)\psi = W(f)\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \psi \ . \tag{3.31}$$

We are now ready to prove the first claim. Let $\psi \in D(\Phi(g))$. Because $\Phi(g)$ is a closed operator and the finite particle vectors are a core of $\Phi(\cdot)$, there exists a sequence $\{\psi_i\}_{i\in\mathbb{N}}$ in $F(\mathcal{H})$ such that $\psi_i \to \psi$ and $\Phi(g)\psi_i \to \Phi(g)\psi$. We will show that the sequence $\Phi(g)W(f)\psi_i$ is Cauchy which implies that $\Phi(g)W(f)\psi_i$ converges to $\Phi(g)W(f)\psi$.

For any $i, j \in \mathbb{N}$, using eq. (3.31) and the unitarity of $W(f)$, we have

$$\begin{aligned} ||\Phi(g)W(f)(\psi_i - \psi_j)|| &= ||W(f)\Phi(g)(\psi_i - \psi_j) - \operatorname{Im}\langle f, g\rangle (\psi_i - \psi_j)|| \\ &\leq ||\Phi(g)(\psi_i - \psi_j)|| + |\operatorname{Im}\langle f, g\rangle|||\psi_i - \psi_j|| \ . \end{aligned} \tag{3.32}$$

Both of the terms on the right of the inequality are Cauchy because they are convergent, so we see that $\Phi(g)W(f)\psi_i$ converges to $\Phi(g)W(f)\psi$ because $\Phi(g)$ is a closed operator. The right hand side of eq. (3.31) converges trivially to $W(f)\Phi(g)\psi - \operatorname{Im}\langle f, g\rangle \psi$ by unitarity and the definition of convergence. We see that eq. (3.31) holds for all $\psi \in D(\Phi(g))$. Finally, let $\psi \in W(f)D(\Phi(g))$, so $\psi = W(f)\phi$ for $\phi \in D(\Phi(g))$. We have

$$||\Phi(g)\psi|| = ||\Phi(g)W(f)\phi|| \leq ||\Phi(g)\phi|| + |\operatorname{Im}\langle f, g\rangle|||\phi|| < \infty \ , \tag{3.33}$$

so $W(f)D(\Phi(g)) \subset D(\Phi(g))$. For the reverse inclusion, let $\psi \in D(\Phi(g))$. Denote $\phi = W(-f)\psi$ so $\psi = W(f)\phi$. By previous the inclusion, we know that $W(-f)\psi \in D(\Phi(g))$, so it follows that $\psi \in W(f)D(\Phi(g))$, and hence we have the equality $W(f)D(\Phi(g)) = D(\Phi(g))$.

2. The most typical, but non-rigorous, proof of this property is given by applying the Baker-Campbell-Haussdorf formula. The problem with this method is that the BCH formula can only be applied in specific circumstances which concern Lie algebras. As we are dealing with unbounded operators, the formula does not directly apply here.

Instead, we will give a variation of a similar proof, and state the details which must be proved to rigorously complete the proof.

Let $f, g \in \mathcal{H}$. We define a function $F : \mathbb{R} \to B(\mathcal{F}^{(+)})$ by

$$F(t) := W(-t(f+g))W(tf)W(tg) \ . \tag{3.34}$$

Note that F is the product of three strongly continuous unitary semi-groups. We will compute the strong derivative of F for $\psi \in F(\mathcal{H})$. To save space, define $x(t) = W(-t(f+g))$, $y(t) = W(tf)$, and $z(t) = W(tg)$. Furthermore, for $h \in \mathbb{R}$, we define $\Delta_h x(t) = x(t+h) - x(t)$. We have the following decomposition

$$\begin{aligned}\frac{F(t+h)-F(t)}{h} &= \frac{\Delta_h x(t)}{h} y(t)z(t) + x(t)\frac{\Delta_h y(t)}{h} z(t) + x(t)y(t)\frac{\Delta_h z(t)}{h} \\ &+ x(t)\frac{\Delta_h y(t)\Delta_h z(t)}{h} + \frac{\Delta_h x(t) y(t)\Delta_h z(t)}{h} + \frac{\Delta_h x(t)\Delta_h y(t)}{h} z(t) \\ &+ \frac{\Delta_h x(t)\Delta_h y(t)\Delta_h z(t)}{h} \ .\end{aligned} \tag{3.35}$$

In order to rigorously take the strong limits, one must show that in all the above cases, the series which are formed by expanding the operators in their power series converge absolutely. In this case, all the terms with two or more Δ_h will necessarily vanish in the limit as $h \to 0$, and we will be left with only strong derivatives. One must also note that proposition 3.1 takes care of the domains of the operators. If one is able to prove these details, then for any $\psi \in F(\mathcal{H})$, we have

$$\begin{aligned}F'(t)\psi &= -i\Phi(f+g)F(t)\psi \\ &+ iW(-t(f+g))\Phi(f)W(tf)W(tg)\psi \\ &+ iW(-t(f+g))W(tf)\Phi(g)W(tg)\psi \ .\end{aligned} \tag{3.36}$$

Using the commutation relations in proposition 3.1, we have

$$\begin{aligned}W(-t(f+g))\Phi(f)W(tf)W(tg)\psi &= \Phi(f)F(t)\psi - \operatorname{Im}\langle -t(f+g), f\rangle F(t)\psi \\ &= (\Phi(f) - t\operatorname{Im}\langle f,g\rangle)F(t)\psi \ ,\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}W(-t(f+g))W(tf)\Phi(g)W(tg)\psi &= W(-t(f+g))\Phi(g)W(tf)W(tg)\psi - t\operatorname{Im}\langle f,g\rangle F(t)\psi \\ &= (\Phi(g) + t\operatorname{Im}\langle f,g\rangle)F(t)\psi - t\operatorname{Im}\langle f,g\rangle F(t)\psi \\ &= \Phi(g)F(t)\psi \ .\end{aligned}$$

Using these two computations, we have

$$F'(t)\psi = -it\operatorname{Im}\langle f,g\rangle F(t)\psi \ . \tag{3.38}$$

This can be written as an integral equation via Bochner integrals. Indeed, we have

$$F(t)\psi - F(0)\psi = -\int_0^t ds\ is\operatorname{Im}\langle f,g\rangle F(s)\psi$$

Let $\phi \in \mathcal{F}^{(+)}$ be arbitrary. We can interchange the order of action with integrals and functionals on $\mathcal{F}^{(+)}$. In particular, we have

$$\langle F(t)\psi, \phi\rangle = \langle F(0)\psi,\phi\rangle - \int_0^t ds\ is\operatorname{Im}\langle f,g\rangle \langle F(s)\psi,\phi\rangle \ .$$

The mapping $t \mapsto \langle F(t)\psi,\phi\rangle$ is a mapping from the real numbers to $\mathbb{C}$, and the unique solution to this differential equation is given by

$$\langle F(t)\psi,\phi\rangle = \langle F(0)\psi,\phi\rangle e^{\frac{-it^2\operatorname{Im}\langle f,g\rangle}{2}} = \left\langle e^{\frac{-it^2\operatorname{Im}\langle f,g\rangle}{2}}\psi,\phi\right\rangle \ . \tag{3.39}$$

Recall that $F(\mathcal{H})$ is dense in $F^{(+)}$, and, by continuity of the inner product and $F(t)$, we see that for any $\eta,\phi \in \mathcal{F}^{(+)}$, we have

$$\langle F(t)\eta,\phi\rangle = \left\langle e^{\frac{-it^2\operatorname{Im}\langle f,g\rangle}{2}}\eta,\phi\right\rangle \ . \tag{3.40}$$

This implies that

$$F(t)\eta = e^{\frac{-it^2\,\mathrm{Im}\langle f,g\rangle}{2}}\eta \ . \tag{3.41}$$

The above holds for all $t \in \mathbb{R}$ and all $\eta \in \mathcal{F}^{(+)}$. Setting $t = 1$

$$W(f)W(g)\eta = e^{\frac{-i\,\mathrm{Im}\langle f,g\rangle}{2}}W(f+g) \tag{3.42}$$

as desired.

3. We will show that the spectrum of $\Phi(\cdot)$ is the whole real-line, and, subsequently, we will use this property along with the spectral theorem to prove the proposition.

Let $t \in \mathbb{R}$. Using eq. (3.18), we have

$$W(itf)\Phi(f)W(-itf)\psi = \Phi(f)\psi - \mathrm{Im}\,\langle itf, f\rangle\,\psi = \Phi(f)\psi - t||f||^2\psi \ , \tag{3.43}$$

for all $\psi \in D(\Phi(f))$. Now, define $h_1 : \sigma(\Phi(f)) \to \mathbb{C}$ and $h_2 : \sigma(\Phi(f)) \to \mathbb{C}$ by

$$h_1(\lambda) = \exp(it\lambda)\lambda\exp(-it\lambda) = \lambda, \ h_2(\lambda) = \lambda - t||f||^2 \ . \tag{3.44}$$

Applying the spectral theorem, for $\psi \in D(\Phi(f))$, we have

$$h_1(\Phi(f))\psi = h_2(\Phi(f))\psi \ . \tag{3.45}$$

Applying the spectral mapping theorem, we have

$$\sigma(h_1(\Phi(f))) = \sigma(h_2(\Phi(g))) \iff \sigma(\Phi(f)) = \sigma(\Phi(f)) - t||f||^2 \ . \tag{3.46}$$

The equation in eq. (3.46) holds for all $t \in \mathbb{R}$. The spectrum is always non-empty, so let $\alpha \in \sigma(\Phi(f))$. Let $\lambda \in \mathbb{R}$, and choose $t = \frac{\alpha-\lambda}{||f||^2}$. By eq. (3.46), we have

$$\alpha - \frac{\alpha-\lambda}{||f||^2}||f||^2 \in \sigma(\Phi(f)) \implies \lambda \in \sigma(\Phi(f) \ . \tag{3.47}$$

We have $\mathbb{R} \subset \sigma(\Phi(f))$, since $\lambda \in \mathbb{R}$ was arbitrary. The opposite inclusion follows since $\Phi(\cdot)$ is self-adjoint, and hence $\sigma(\Phi(f)) = \mathbb{R}$.

Next, consider the spectral representation of $W(f)$. We have

$$W(f) = \int_{\sigma(\Phi(f))} dE(\lambda)e^{i\lambda} = \int_{\mathbb{R}} dE(\lambda)e^{i\lambda} \ . \tag{3.48}$$

For $\psi \in \mathcal{F}$, we compute

$$||W(f)\psi - \psi||^2 = \int_{\mathbb{R}} d\,\langle\psi, E(\lambda)\psi\rangle\,\left|e^{i\lambda}-1\right|^2 \tag{3.49}$$

$$= 2\int_{\mathbb{R}} d\,\langle\psi, E(\lambda)\psi\rangle\,(1-\cos\lambda) \ . \tag{3.50}$$

Now, let $\delta > 0$ be arbitrary, and choose $\psi_\delta \in \mathcal{F}$ such that $E([\pi-\delta, \pi+\delta])\psi_\delta = \psi_\delta$ and $||\psi_\delta|| = 1$. Note that such a ψ_δ always exists. Indeed, if it does not exist then by the spectral theorem the mapping $\Phi(f) - \pi\mathbb{1}$ would be invertible, but the spectrum of $\Phi(f)$ is the whole real line which is a contradiction. By continuity, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$1 - \cos\lambda - 2 > -\varepsilon \tag{3.51}$$

for all $\lambda \in [-\pi-\delta, \pi+\delta]$. Let $\varepsilon > 0$ be arbitrary, let ψ_δ be as defined earlier for the δ which we obtain from continuity. We compute

$$\begin{aligned} ||W(f)\psi_\delta - \psi_\delta||^2 &= 2\int_{\mathbb{R}} d\,\langle \psi_\delta, E(\lambda)\psi_\delta\rangle\,(1-\cos\lambda - 2 + 2) \\ &= 4 + 2\int_{\mathbb{R}} d\,\langle \psi_\delta, E(\lambda)\psi_\delta\rangle\,(1-\cos\lambda - 2) \\ &\geq 4 - 2\varepsilon \int_{[-\pi-\delta,\pi+\delta]} d\,\langle \psi_\delta, E(\lambda)\psi_\delta\rangle \\ &= 4 - 2\varepsilon\ . \end{aligned} \tag{3.52}$$

Since $\varepsilon > 0$ was arbitrary, we have

$$2 \leq ||W(f) - \mathbb{1}|| \leq ||W(f)|| + ||\mathbb{1}|| = 2\ . \tag{3.53}$$

Thus, $||W(f) - \mathbb{1}|| = 2$, as desired. $\square$

4 Abstract Weyl Operators and the C^*-algebraic Structure

The commutation relations given in eq. (3.19) are called the Weyl relations, and the C^*-algebra generated by the Weyl operators, as defined above, is called the CCR-algebra on $\mathcal{H}$.

We will, for the moment, work in higher generality and then remark on the previous case. Let H be a real vector space equipped with a symplectic from σ. By symplectic form, we mean that $\sigma : H \times H \to \mathbb{R}$ such that σ is bilinear, $\sigma(f,f) = 0$ for all $f \in H$, and $\sigma(f,g) = 0$ for all $g \in H$ implies that $f = 0$. Let $\{W(f) : f \in H\}$ be a collection of elements of a Banach algebra with identity which satisfy

$$W(-f) = W^*(f) \tag{4.1}$$

and

$$W(f)W(g) = e^{i\sigma(f,g)}W(f+g)\ , \tag{4.2}$$

for any $f, g \in H$. We have

$$W(f)W^*(f) = W(f)W(-f) = e^{i\sigma(f,-f)}W(f-f) = e^{i\sigma(-f,f)}W(-f+f) = W^*(f)W(f)\ . \tag{4.3}$$

The C^*-algebra generated by the collection of elements $\{W(f) : f \in H\}$ is a C^*-algebra with identity because

$$W(0)W(f) = e^{i\sigma(0,f)}W(0+f) = W(f) = e^{i\sigma(f,0)}W(f+0) = W(f)W(0)\ . \tag{4.4}$$

This shows that $W(0)$ is the identity.

Denote the C^*-algebra generated by the collection of elements $\{W(f) : f \in H\}$ by $\mathcal{B}$ and consider the real linear space H as an Abelian group with vector addition as its binary operation. Then, for any $f, g \in H$, we have

$$W(f+g) = e^{-i\sigma(f,g)}W(f)W(g)\ . \tag{4.5}$$

This computation shows that the binary operation in H is mapped to a scalar multiplier of the corresponding multiplication in the algebra. With this observation, we claim that

$$\mathcal{B} = \overline{\left\{ \sum_{i=1}^{n} \lambda_i W(f_i) : \lambda_i \in \mathbb{C},\ f_i \in H \right\}}\ . \tag{4.6}$$

The following is a sketch of this simple but notationally tedious proof. By definition, $\mathcal{B}$ is the smallest algebra which contains all multi-variable polynomials of finite degree with the generators of $\mathcal{B}$ as the variables. By the earlier observation regarding the interplay of the linear and multiplicative structure via the operator $W(\cdot)$, we have

$$\prod_{i=1}^{n}(W(f_i))^{k_i} = C(f_1,\ ...,\ f_n, k_1,\ ...,\ k_n)W\left(\sum_{i=1}^{n} k_i f_i\right) \ . \tag{4.7}$$

In the above, $C(\cdot)$ is a complex number that depends upon the given variables. This observation shows that polynomials of the generators are mapped to linear combinations of the space. Thus, we can swap the set of linear combinations of the generators by finite index multi-variable polynomials of the generators, and the claim follows.

This interplay between addition in the linear space and multiplication in the algebra will be crucial in the upcoming proofs. It will allow us to construct $*$-isomorphisms by only examining linear structure, and avoiding the (possibly) problematic explicit construction of the isomorphisms.

If $\mathcal{H}$ is a complex Hilbert space, then it can also be regarded as real linear space by swapping the field of complex numbers to the field of real numbers. Then we can consider the symplectic form $\sigma(f,g) = -\frac{1}{2}\operatorname{Im}\langle f, g\rangle$ and we see that the collection of operators defined in proposition 3.1 satisfy the algebraic relations we have given.

In this way, we see that the purely algebraic formalism is an abstraction of the framework of operators we previously defined. However, it will be shown that this abstraction does not generate anything "new", so to speak. In fact, it will be shown that all C^*-algebras which are generated by elements which satisfy the Weyl relations for some real linear space H and symplectic form σ are mutually $*$-isomorphic.

4.1 Existence of Abstract C^*-algebras Generated by Weyl Operators

First, we must show that such C^*-algebras exist in general. So far, we have dealt with the concrete symplectic form given by the imaginary part of the inner product. The following theorem will prove the existence of these C^*-algebras.

Theorem 4.1. *Let H be a real linear space and σ a symplectic form on H. Then there exists a C^*-algebra generated by a collection of operators $\{R_a : a \in H\}$ which satisfy*

$$R_a^* = R_{-a} \tag{4.8}$$

and

$$R_a R_b = e^{i\sigma(a,b)} R_{a+b} \ . \tag{4.9}$$

Proof. The idea of this proof will be to use the linear structure of H to define a suitable Hilbert space. Then we will construct unitary operators on this Hilbert space and show that these unitary operators have the desired properties.

Initially, the real linear space H has no topological structure. We can endow the space H with the discrete topology and consider H to be a group with vector addition as its binary operator. From this perspective, we can view H as a discrete Abelian group. Define the space

$$\ell_2(H) = \left\{\psi \in \mathbb{C}^H : \sum_{x\in H} |\psi(x)|^2 < \infty \right\} \ . \tag{4.10}$$

The space $\ell_2(H)$ is actually the space $L^2(H, 2^H, \mu)$ where 2^H is the collection of all subsets of H which forms a σ-algebra and μ is the point counting measure. From these observations, it is clear that $\ell_2(H)$ is a Hilbert space.

Next, let $y \in H$ be arbitrary, and define a mapping $R_y : \ell_2(H) \to \ell_2(H)$ by

$$(R_y\psi)(x) = e^{i\sigma(x,y)}\psi(x+y) \ . \tag{4.11}$$

This mapping is obviously linear, and, for a linear space $H - y = H$, thus the mapping is well-defined since

$$\sum_{x\in H} |(R_y\psi)(x)|^2 = \sum_{x\in H} |\psi(x+y)|^2 = \sum_{x+y\in H} |\psi(x+y)|^2 = \sum_{x\in H} |\psi(x)|^2 < \infty \ . \tag{4.12}$$

Let $\psi, \phi \in \ell_2(H)$. We have

$$\begin{aligned}\left\langle R_y^*\phi, \psi\right\rangle = \langle \phi, R_y\psi\rangle &= \sum_{x\in H} \phi(x)^* e^{i\sigma(x,y)}\psi(x+y) && (4.13)\\ &= \sum_{x\in H} \left(\phi(x-y)e^{-i\sigma(x-y,y)}\right)^* \psi(x)\\ &= \sum_{x\in H} \left(\phi(x-y)e^{-i\sigma(x,y)}\right)^* \psi(x) \ .\end{aligned}$$

$$\tag{4.14}$$

In the above calculation, we used the fact that $\sigma(y, y) = 0$ for all $y \in H$. From the above calculation, we see that

$$(R_y^*\psi)(x) = e^{-i\sigma(x,y)}\psi(x-y) = (R_{-y}\psi)(x) \ . \tag{4.15}$$

Finally, we compute

$$(R_y R_y^*\psi)(x) = e^{-i\sigma(x,y)}(R_y\psi)(x-y) = e^{-i\sigma(x,y)}e^{i\sigma(x-y,y)}\psi(x-y+y) = \psi(x) \ , \tag{4.16}$$

with the same computation, one shows that $(R_y^* R_y\psi)(x) = \psi(x)$. Thus, R_y is a unitary operator for any $y \in \mathcal{H}$. Let $a, b \in H$, we have

$$\begin{aligned}(R_a R_b\psi)(x) = e^{i\sigma(x,a)}(R_b\psi)(x+a) &= e^{i\sigma(x,a)}e^{i\sigma(x+a,b)}\psi(x+a+b) && (4.17)\\ &= e^{i\sigma(a,b)}e^{i\sigma(x,a+b)}\psi(x+a+b) = e^{i\sigma(a,b)}(R_{a+b}\psi)(x) \ .\end{aligned}$$

By the above, we have

$$R_a R_b = e^{i\sigma(a,b)} R_{a+b} \ . \tag{4.18}$$

By eq. (4.15) and eq. (4.18), we see that the collection of operators $\{R_a : a \in H\}$ satisfy

$$R_a^* = R_{-a} \tag{4.19}$$

and

$$R_a R_b = e^{i\sigma(a,b)} R_{a+b} \ . \tag{4.20}$$

The collection of operators $\{R_a : a \in H\}$ generate a C^*-algebra, and the elements R_a satisfy the relations given in the theorem. $\square$

4.2 C^*-algebraic Uniqueness

By the previous construction, for every real linear space H and symplectic form σ there exists a C^*-algebra generated by elements which satisfy the Weyl-relations. Next, we will show all such C^*-algebras are $*$-isomorphic.

Theorem 4.2. *Let H be a real linear space and σ a symplectic form on H. For $i = 1, 2$, let $\mathcal{H}_i$ be separable Hilbert spaces and $\mathcal{B}_i \subset B(\mathcal{H}_i)$ closed sub-algebras of bounded operators of the spaces $\mathcal{H}_i$ which are generated by the collections of operators $\{W_i(f) : f \in H\}$ which satisfy the Weyl relations.*

Then $\mathcal{B}_1$ is $*$-isomorphic to $\mathcal{B}_2$, and, furthermore, this isomorphism which we will denote by α is the unique $*$-isomorphism such that

$$\alpha(W_1(f)) = W_2(f) \tag{4.21}$$

for all $f \in H$.

Proof. With reference to [12, p. 40], define the Hilbert space

$$\ell_2(H, \mathcal{H}_i) = \left\{ \psi \in \mathcal{H}_i^H : \sum_{x \in H} ||\psi(x)||^2 < \infty \right\} . \tag{4.22}$$

The elements of $\ell_2(H, \mathcal{H}_i)$ are occasionally cumbersome to work with, so we will instead utilize the natural isomorphism $\ell_2(H, \mathcal{H}_i) \cong \ell_2(H) \bigotimes \mathcal{H}_i$ from [12, p. 52]. Denote this isomorphism by T_i, and note that for any $\psi \in \ell_2(H)$ and $\phi \in \mathcal{H}_i$, we have

$$T_i(\psi \otimes \phi)(x) = \psi(x)\phi . \tag{4.23}$$

Let $y \in H$ and define $\pi_{y,i} : \ell_2(H, \mathcal{H}_i) \to \ell_2(H, \mathcal{H}_i)$ by

$$(\pi_{y,i}\psi)(x) = W_i(y)\psi(x + y) . \tag{4.24}$$

By unitarity of $W_i(y)$, we have

$$||(\pi_{y,i}\psi)(x)|| = ||W_i(y)\psi(x + y)|| = ||\psi(x + y)|| , \tag{4.25}$$

and

$$||\pi_{y,i}\psi||^2 = \sum_{x \in H} ||(\pi_{y,i}\psi)(x)||^2 = \sum_{x \in H} ||\psi(x + y)||^2 = \sum_{x \in H} ||\psi(x)||^2 = ||\psi||^2 .$$

This calculation shows that the operator $\pi_{y,i}$ is an isometry. Furthermore, let $\phi \in \ell_2(H, \mathcal{H}_i)$ and define $\psi(x) = W_i(-y)\phi(x - y)$. By the same calculation as above, we have $\psi \in \ell_2(H, \mathcal{H}_i)$, and

$$(\pi_{y,i}\psi)(x) = W_i(y)\psi(x + y) = W_i(y)W_i(-y)\phi(x + y - y) = \phi(x) . \tag{4.26}$$

This shows that $\pi_{y,i}$ is a surjection. By virtue of being a surjective isometry, the operator $\pi_{y,i}$ is a unitary operator.

Define the operator $U_i : \ell_2(H, \mathcal{H}_i) \to \ell_2(H, \mathcal{H}_i)$ by

$$(U_i\psi)(x) = W_i(x)\psi(x) . \tag{4.27}$$

The operator U_i is a unitary operator by the same computations as for the operators $\pi_{y,i}$.

Let $y \in H$, and define the operator $R_y \times \mathbb{1}_i : \ell_2(H, \mathcal{H}_i) \to \ell_2(H, \mathcal{H}_i)$ by

$$((R_y \times \mathbb{1}_i)\psi)(x) = e^{i\sigma(x,y)}\psi(x + y) . \tag{4.28}$$

Once again, by largely the same computations as for the operators U_i and $\pi_{y,i}$, the operator $R_y \times \mathbb{1}_i$ is a unitary operator.

Using these operators, for any $y \in H$, we have

$$\begin{aligned}((U_i\pi_{y,i})\psi)(x) = W_i(x)(\pi_{y,i}\psi)(x) &= W_i(x)W_i(y)\psi(x+y) \\ &= e^{i\sigma(x,y)}W_i(x+y)\psi(x+y) \\ &= e^{i\sigma(x,y)}(U_i\psi)(x+y) \\ &= (((R_y \times \mathbb{1}_i)U_i)\psi)(x)\ .\end{aligned} \tag{4.29}$$

This computation shows that

$$U_i\pi_{y,i}U_i^* = R_y \times \mathbb{1}_i\ . \tag{4.30}$$

The operators $\pi_{y,i}$ and $R_y \times \mathbb{1}_i$ are thus equivalent.

Next, we will justify the notation of the operator $R_y \times \mathbb{1}_i$. Define

$$D_i = \{\psi \otimes \phi : \psi \in \ell_2(H),\ \phi \in \mathcal{H}_i\}\ , \tag{4.31}$$

and note that $\overline{\operatorname{span}(D_i)} = \ell_2(H) \otimes \mathcal{H}_i$. Using the natural isomorphism T_i, we compute

$$\begin{aligned}(((R_y \times \mathbb{1}_i)T_i)(\psi \otimes \phi))(x) &= e^{i\sigma(x,y)}(T_i(\psi \otimes \phi))(x+y) \\ &= e^{i\sigma(x,y)}\psi(x+y)\phi \\ &= (T_i((R_y \otimes \mathbb{1}_i)(\psi \otimes \phi))(x)\ .\end{aligned} \tag{4.32}$$

The above computation holds for any $\psi \in \ell_2(H)$ and $\phi \in \mathcal{H}_i$. Since $\operatorname{span}(D_i)$ was dense in $\ell_2(H) \otimes \mathcal{H}_i$, and the bounded linear extension is unique, we have

$$T_i^*(R_y \times \mathbb{1}_i)T_i = R_y \otimes \mathbb{1}_i\ . \tag{4.33}$$

The operators $R_y \times \mathbb{1}_i$ and $R_y \otimes \mathbb{1}_i$ are thus equivalent.

We will show that the operators $R_y \otimes \mathbb{1}_1$ and $R_y \otimes \mathbb{1}_2$ are equivalent.

We note that $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic because they are both separable Hilbert spaces. Denote the isomorphism between these two Hilbert spaces by V. For any $\psi \in \ell_2(H)$ and $\phi \in \mathcal{H}_i$, we have

$$(\mathrm{id} \otimes V)(R_y \otimes \mathbb{1}_1)(\psi \otimes \phi) = R_y\psi \otimes V\phi = (R_y \otimes \mathbb{1}_2)(\mathrm{id} \otimes V)(\phi \otimes \psi)\ . \tag{4.34}$$

The operator $\mathrm{id} \otimes V$ is unitary by virtue of being a tensor product of two unitary operators, and, again, applying extension by linearity from a dense set, we have

$$(\mathrm{id} \otimes V)(R_y \otimes \mathbb{1}_1)(\mathrm{id} \otimes V)^* = R_y \otimes \mathbb{1}_2\ . \tag{4.35}$$

We can neatly summarize the equivalences with the following diagram

$$\pi_{y,1} \sim R_y \times \mathbb{1}_1 \sim R_y \otimes \mathbb{1}_1 \sim R_y \otimes \mathbb{1}_2 \sim R_y \times \mathbb{1}_2 \sim \pi_{y,2}\ . \tag{4.36}$$

The end result is that the operators $\pi_{y,1}$ and $\pi_{y,2}$ are equivalent. It remains to show that the C^*-algebra generated by the collection of elements $\{\pi_{y,i} : y \in H\}$, which we will denote by Π_i, is $*$-isomorphic to $\mathcal{B}_i$.

A simple verification, which follows the same steps as the proof in theorem 4.1, shows that the generators of Π_i satisfy the Weyl relations. By earlier remarks regarding the interplay between the linear structure of H and the multiplicative structure of the algebras in relation to the Weyl operators, we know that

$$\Pi_i = \overline{\left\{ \sum_{k=1}^{n} \lambda_k \pi_{y_k,i} : \lambda_k \in \mathbb{C},\ y_k \in H \right\}}, \mathcal{B}_i = \overline{\left\{ \sum_{k=1}^{n} \lambda_k W_i(f_k) : \lambda_k \in \mathbb{C},\ f_k \in H \right\}} . \tag{4.37}$$

In order to show that these algebras are *-isomorphic, it is enough to show that

$$\left\| \sum_{k=1}^{n} \lambda_k \pi_{y_k,i} \right\| = \left\| \sum_{k=1}^{n} \lambda_k W_i(y_k) \right\| . \tag{4.38}$$

This would imply that the natural mapping, which maps a generator to the corresponding generator of the other algebra, is a linear isometry on a dense subset, and thus we can linearly extend it to a unitary mapping on the whole space.

In the proof of theorem 4.1, we regarded H as a discrete Abelian group. We will once again rely on this interpretation. Because H is equipped with the discrete topology, it is obviously locally compact and Haussdorf. The space $\ell_2(H)$ was defined via an integral with the point counting measure. The point counting measure is an example of a Haar measure on H. Denote μ to be the point counting measure on H, then there exists a unique up-to-factor Haar measure ν on the dual group $\hat{H}$ such that the Fourier transform $\mathcal{F} : \ell_2(H) \to L^2(\hat{H})$, which is defined by

$$(\mathcal{F}\psi)(\chi) = \int_H d\mu(x)\psi(x)\chi^*(x) \tag{4.39}$$

is an isomorphism.

Let $y \in H$ and define an operator $\hat{\pi}_{y,i} : L^2(\hat{H}, \mathcal{H}_i) \to L^2(\hat{H}, \mathcal{H}_i)$ by

$$(\hat{\pi}_{y,i}\psi)(\chi) = W_i(y)\chi(y)\psi(\chi) . \tag{4.40}$$

This operator is obviously an isometry since the dual group consists of characters which satisfy $|\chi(y)| = 1$. Note that for any $\chi \in \hat{H}$, we have $\chi H \subset \mathbb{D}$. This implies that $\chi(y) \neq 0$ for any $y \in H$, and for any $\phi \in L^2(\hat{H}, \mathcal{H}_i)$, we can define $\psi(\chi) = W_i(-y)(\chi(y))^{-1}\phi(\chi)$. We have

$$(\hat{\pi}_{y,i}\psi)(\chi) = W_i(y)\chi(y)\psi(\chi) = W_i(y)W_i(-y)\chi(y)(\chi(y))^{-1}\phi(\chi) = \phi(\chi) . \tag{4.41}$$

This shows that $\hat{\pi}_{y,i}$ is a surjection and thus a unitary operator.

The small hat in $\hat{\pi}_{y,i}$ is suggestive notation for the relationship between the operator $\pi_{y,i}$ and an equivalent operator in the space $L^2(\hat{H}, \mathcal{H}_i)$. This is indeed the case, as we shall next show that these operators are equivalent with an application of the Fourier transform.

Define

$$\mathcal{D}_i = \{\psi \otimes \phi : \psi \in L^2(\hat{H}),\ \phi \in \mathcal{H}_i\} . \tag{4.42}$$

As before, we have $\overline{\mathrm{span}(\mathcal{D}_i)} = L^2(\hat{H}) \otimes \mathcal{H}_i$. Denote $\mathcal{T}_i$ to be natural isomorphism between $L^2(\hat{H}) \otimes \mathcal{H}_i$ and $L^2(\hat{H}, \mathcal{H}_i)$. Define a multiplication operator M_y by

$$(M_y\psi)(\chi) = \psi(\chi)\chi(y) . \tag{4.43}$$

We compute

$$\begin{aligned}(\hat{\pi}_{y,i}\mathcal{T}_i(\psi \otimes \phi))(\chi) = W_i(y)\chi(y)T(\psi \otimes \phi)(\chi) &= \psi(\chi)\chi(y)W_i(y)\phi \\ &= (\mathcal{T}_i(M_y \otimes W_i(y))(\psi \otimes \phi))(\chi) .\end{aligned} \tag{4.44}$$

In the above, we have used the multiplication operator by M_y. Because we are multiplying by a character, the multiplication operator is obviously unitary. Again, applying extension by linearity, we have

$$\mathcal{T}_i^* \hat{\pi}_{y,i} \mathcal{T}_i = M_y \otimes W_i(y) \ . \tag{4.45}$$

The operators $\hat{\pi}_{y,i}$ and $M_y \otimes W_i(y)$ are thus equivalent.

Denote $\mathcal{F} : \ell_2(H) \to L^2(\hat{H})$ to be the Fourier transform. The following equivalence is based on the fact that the Fourier transform maps the group operation on H to multiplication of characters in $L^2(\hat{H})$. To be more explicit, let $\psi \in L^2(\hat{H})$, we have

$$\begin{aligned}(\mathcal{F}^{-1} M_y \psi)(x) = \int_{\hat{H}} d\nu(\chi)(M_y\psi)(\chi)\chi(x) &= \int_{\hat{H}} d\nu(\chi)\psi(\chi)\chi(x)\chi(y) \\ &= \int_{\hat{H}} d\nu(\chi)\psi(\chi)\chi(x+y) \\ &= (\mathcal{F}^{-1}\psi)(x+y) \ .\end{aligned} \tag{4.46}$$

Finally, let $\psi \in L^2(\hat{H})$ and $\phi \in \mathcal{H}_i$, we have

$$\begin{aligned}(T_i((\mathcal{F}^{-1} \otimes \mathbb{1}_i)(M_y \otimes W_i(y))(\psi \otimes \phi)))(x) &= (\mathcal{F}^{-1} M_y \psi)(x) W_i(y)\phi && (4.47) \\ &= (\mathcal{F}^{-1}\psi)(x+y) W_i(y)\phi \\ &= ((\pi_{y,i}(T_i(\mathcal{F}^{-1} \otimes \mathbb{1}_i)(\psi \otimes \phi)))(x) \ . && (4.48)\end{aligned}$$

Applying extension by linearity, we have

$$T_i(\mathcal{F}^{-1} \otimes \mathbb{1})(M_y \otimes W_i(y))(T_i(\mathcal{F}^{-1} \otimes \mathbb{1}))^* = \pi_{y,i} \ . \tag{4.49}$$

We summarize the proof with the following diagram

$$\hat{\pi}_{y,i} \sim M_y \otimes W_i(y) \sim \pi_{y,i} \ . \tag{4.50}$$

The operators $\hat{\pi}_{y,i}$ and $\pi_{y,i}$ are thus equivalent.

Applying the unitary operator which yields this equivalence, we have

$$\left\| \sum_{k=1}^{n} \lambda_k \pi_{y_k,i} \right\| = \left\| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right\| \ . \tag{4.51}$$

Our next goal is to write the norm on the right hand side of the above equality in a more tractable form. Note that $\hat{\pi}_{y,i}$ corresponds to a multiplication operator, we will show that

$$\left\| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right\| = \sup_{\chi \in \hat{H}} \left\| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right\| \ . \tag{4.52}$$

First, we remark that when taking the operator norm of a bounded operator, the supremum which appears in the norm can be taken over any dense set of the initial space. Recall that $\mathrm{span}(\mathcal{D}_i)$ was dense in $L^2(\hat{H}) \otimes \mathcal{H}_i$, since $\mathcal{T}_i$ was an isomorphism between $L^2(\hat{H}) \otimes \mathcal{H}_i$ and $L^2(\hat{H}, \mathcal{H}_i)$, it follows that

$$L^2(\hat{H}, \mathcal{H}_i) = \mathcal{T}_i(L^2(\hat{H}) \otimes \mathcal{H}_i) = \mathcal{T}_i(\overline{\mathrm{span}(\mathcal{D}_i)}) = \overline{\mathrm{span}(\mathcal{T}_i(\mathcal{D}_i))} \ . \tag{4.53}$$

We see that the space $\mathrm{span}(\mathcal{T}_i(\mathcal{D}_i))$ is dense in $L^2(\hat{H}, \mathcal{H}_i)$, and, if $\Psi \in \mathrm{span}(\mathcal{T}_i(\mathcal{D}_i))$, then there exists $J \in \mathbb{N}$, $\psi_j \in L^2(\hat{H})$, $\phi_j \in \mathcal{H}_i$ and $a_j \in \mathbb{C}$ such that

$$\Psi(\chi) = \sum_{j=1}^{J} a_j \psi_j(\chi) \phi_j \ . \tag{4.54}$$

One notes that without loss of generality, we can choose the ϕ_j to be orthonormal, and include the a_j in the ψ_j. Because of the orthonormaltiy, we will have

$$\Psi(\chi) = \sum_{j=1}^{J} \psi_j(\chi)\phi_j \ , \tag{4.55}$$

and

$$||\Psi||^2 = \sum_{j=1}^{J} ||\psi_j||^2 \ . \tag{4.56}$$

Using these observations, for such a Ψ as above, we have

$$\begin{aligned} \left|\left|\sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \Psi\right|\right|^2 &= \int_{\hat{H}} d\nu(\chi) \left|\left|\sum_{k=1}^{n}\sum_{j=1}^{J} \lambda_k \psi_j(\chi)\chi(y_k)W_i(y_k)\phi_j\right|\right|^2 \\ &\leq \sum_{j=1}^{J} \int_{\hat{H}} d\nu(\chi) \left|\left|\sum_{k=1}^{n} \lambda_k \psi_j(\chi)\chi(y_k)W_i(y_k)\phi_j\right|\right|^2 \ . \end{aligned} \tag{4.57}$$

For any ψ_j and ϕ_j as above, we compute

$$\begin{aligned} \int_{\hat{H}} d\nu(\chi) \left|\left|\sum_{k=1}^{n} \lambda_k \psi_j(\chi)\chi(y_k)W_i(y_k)\phi_j\right|\right|^2 &= \int_{\hat{H}} d\nu(\chi)|\psi_j(\chi)|^2 \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\phi_j\right|\right|^2 \\ &\leq ||\phi_j||^2 \int_{\hat{H}} d\nu(\chi)|\psi_j(\chi)|^2 \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\right|\right|^2 \\ &= \int_{\hat{H}} d\nu(\chi)|\psi_j(\chi)|^2 \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\right|\right|^2 \ . \end{aligned} \tag{4.58}$$

Combing this with the previous result, we have

$$\left|\left|\sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \Psi\right|\right|^2 \leq \sum_{j=1}^{J} \int_{\hat{H}} d\nu(\chi)|\psi_j(\chi)|^2 \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\right|\right|^2 \ . \tag{4.59}$$

We will show that the mapping

$$\chi \mapsto \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\right|\right| : \hat{H} \to \mathbb{R}^+ \tag{4.60}$$

is continuous. Recall that the dual group $\hat{H}$ is a LCA group when equipped with the inherited topology of pointwise convergence. In fact, we have a stronger result. Because H is a discrete Abelian group, $\hat{H}$ is a compact group.

We will now show that the aforementioned mapping is continuous. Let $\{\chi_\lambda\}_{\lambda\in\Lambda}$ be a convergent net in $\hat{H}$ such that $\chi_\lambda \to \chi$. In this topology, a net converges if and only if $\chi_\lambda(y) \to \chi(y)$ for each $y \in H$. We have

$$\left|\left|\sum_{k=1}^{n} \lambda_k \chi_\lambda(y_k)W_i(y_k) - \sum_{k=1}^{n} \lambda_k \chi(y_k)W_i(y_k)\right|\right| \leq \sum_{k=1}^{n} |\lambda_k||\chi_\lambda(y_k) - \chi(y_k)| \ . \tag{4.61}$$

In the inequality, we used the fact that $W_i(y_k)$ are unitary operators. Since we have convergence in net, the above inequality shows that we also have convergence in nets in the target space. This shows that the mapping

$$\chi \mapsto \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \tag{4.62}$$

is continuous and the mapping which takes an element of a Hilbert space to its norm is always continuous. By virtue of being the composition of two continuous functions, we see that the mapping

$$\chi \mapsto \left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right|\right| \tag{4.63}$$

is continuous. This mapping is thus a continuous mapping from a compact set $\hat{H}$ to the set of positive real numbers, and thus the maximum of this mapping exists and is achieved for some $\tilde{\chi} \in \hat{H}$. Explicitly, for all $\chi \in \hat{H}$, we have

$$\left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right|\right| \leq \left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right| \ . \tag{4.64}$$

Applying this bound, we have

$$\int_{\hat{H}} d\nu(\chi) \sum_{j=1}^{J} |\psi_j(\chi)|^2 \left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right|\right|^2 \leq \int_{\hat{H}} d\nu(\chi) \sum_{j=1}^{J} |\psi_j(\chi)|^2 \left(\left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right|^2 \right) \tag{4.65}$$

$$\begin{aligned} &= \sum_{j=1}^{J} ||\psi_j||^2 \left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right|^2 \\ &= \left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right|^2 ||\Psi||^2 \ . \end{aligned} \tag{4.66}$$

This computation shows that

$$\left|\left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \Psi \right|\right|^2 \leq \left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right|^2 ||\Psi||^2 \ . \tag{4.67}$$

Because we choose Ψ from a dense set, this computation shows that we have the following bound for the operator norm

$$\left|\left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right|\right| \leq \left|\left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right| = \sup_{\chi \in \hat{H}} \left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right|\right| \ . \tag{4.68}$$

Next, we will show that

$$\sup_{\chi \in \hat{H}} \left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right|\right| \leq \left|\left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right|\right| \ . \tag{4.69}$$

First, let $\varepsilon > 0$. By continuity, there exists a neighbourhood U_ε of $\tilde{\chi}$ such that for $\chi \in U_\varepsilon$, we have

$$\left|\left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) - \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right|\right| \leq \varepsilon \ . \tag{4.70}$$

Applying this inequality, for $\phi \in \mathcal{H}_i$ such that $||\phi|| = 1$ and $\chi \in U_\varepsilon$, we have

$$\left| \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \phi \right| \right| - \left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \phi \right| \right| \right| \leq \varepsilon \ . \tag{4.71}$$

By definition of the operator norm, there exists an element $\phi_\varepsilon \in \mathcal{H}_i$ such that $||\phi_\varepsilon|| = 1$, and

$$\left| \left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \phi_\varepsilon \right| \right| - \left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right| \right| \right| \leq \varepsilon \tag{4.72}$$

Combining all of the above inequalities, for $\chi \in U_\varepsilon$, we have

$$\left| \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \phi_\varepsilon \right| \right| - \left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right| \right| \right| \leq 2\varepsilon \ . \tag{4.73}$$

Next, define $\psi_\varepsilon \in L^2(\hat{H})$ by

$$\psi_\varepsilon(x) = \frac{\mathbb{1}(U_\varepsilon)(x)}{\sqrt{\nu(U_\varepsilon)}} \ . \tag{4.74}$$

The Haar measure ν is non-trivial, and thus the measure of any non-empty open set is non-zero. By these observations, the the above element of $L^2(\hat{H})$ is well-defined and $||\psi_\varepsilon|| = 1$.

Finally, applying the previous two relations, we have

$$\begin{aligned} \int_{\hat{H}} d\nu(\chi) |\psi_\varepsilon(\chi)|^2 \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \phi_\varepsilon \right| \right|^2 &= \frac{1}{\nu(U_\varepsilon)} \int_{U_\varepsilon} d\nu(\chi) \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \phi_\varepsilon \right| \right|^2 \\ &\geq \left(\left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right| \right| - 2\varepsilon \right)^2 \ . \end{aligned} \tag{4.75}$$

The elements $\psi_\varepsilon \in L^2(\hat{H})$ and $\phi_\varepsilon \in \mathcal{H}_i$ satisfy $||\psi_\varepsilon|| ||\phi_\varepsilon|| = 1$. We thus have

$$\left| \left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right| \right|^2 \geq \int_{\hat{H}} d\nu(\chi) |\psi_\varepsilon(\chi)|^2 \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \phi_\varepsilon \right| \right|^2 \geq \left(\left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right| \right| - 2\varepsilon \right)^2 \ . \tag{4.76}$$

This inequality holds for arbitrary $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have

$$\left| \left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right| \right| \geq \left| \left| \sum_{k=1}^{n} \lambda_k \tilde{\chi}(y_k) W_i(y_k) \right| \right| = \sup_{\chi \in \hat{H}} \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right| \right| \ . \tag{4.77}$$

Since we have the inequality in both directions, we have equality, and we thus have

$$\left| \left| \sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i} \right| \right| = \sup_{\chi \in \hat{H}} \left| \left| \sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k) \right| \right| \ , \tag{4.78}$$

as desired.

The supremum on the right hand side of the above equation is taken over the dual group $\hat{H}$, however, as in the case of the operator norm, we can replace the dual group with any dense subset of $\hat{H}$. Define

$$\hat{G} = \left\{ \chi \in \mathbb{C}^H : x \in H, \ \chi(y) = e^{2i\sigma(x,y)} \right\} \ . \tag{4.79}$$

We equip the set $\hat{G}$ with the same group operation as the dual group $\hat{H}$. To show that this is a subgroup, it is enough to show that if $\chi_1, \chi_2 \in \hat{G}$ then $\chi_1(\chi_2)^{-1} \in \hat{G}$. There exists $x_1, x_2 \in H$ such that $\chi_1(y) = e^{2i\sigma(x_1,y)}$ and $\chi_2(y) = e^{2i\sigma(x_2,y)}$. It is elementary to check that $(\chi_2)^{-1}(y) = e^{2i\sigma(-x_2,y)}$, and that $(\chi_1\chi_2^{-1})(y) = e^{2i\sigma(x_1-x_2,y)}$. This shows that $\chi_1(\chi_2)^{-1} \in \hat{G}$, so $\hat{G}$ is a subgroup.

The closure of a subgroup is again a subgroup. To see this, recall that the dual group $\hat{H}$ is a topological group in which inversion and the binary operation are continuous mappings. Define $F : \hat{G} \times \hat{G} \to \hat{G}$ by

$$F(\chi_1, \chi_2) = \chi_1\chi_2^{-1} \ . \tag{4.80}$$

In the inherited topology from $\hat{H}$ this is a continuous mapping. Using the fact that we are in the product topology, and the continuity of F, we have

$$F(\overline{\hat{G}} \times \overline{\hat{G}}) = F(\overline{\hat{G} \times \hat{G}}) \subset \overline{F(\hat{G} \times \hat{G})} \subset \overline{\hat{G}} \ . \tag{4.81}$$

This shows that for $\chi_1, \chi_2 \in \overline{\hat{G}}$, we have $\chi_1\chi_2^{-1} \in \overline{\hat{G}}$ which implies that $\overline{\hat{G}}$ is a subgroup.

Since $\hat{G} \subset \hat{H}$ and $\hat{H}$ is compact, it follows that $\overline{\hat{G}}$ is a closed subgroup of $\hat{H}$. We will show that $\hat{G}$ is, in fact, dense in $\hat{H}$. To do this, we will need a result in abstract harmonic analysis. We will utilize the corollary from [6, p. 366, Corollary 23.26].

Suppose that $\overline{\hat{G}}$ is a proper subset of $\hat{H}$. This implies that there exists an element $\chi \in \hat{H}$ such that $\chi \notin \overline{\hat{G}}$. By the given corollary, there exists an element $\hat{\chi} \in \hat{\hat{H}}$ such that $\hat{\chi}\hat{G} = \{1\}$ but $\hat{\chi}(\chi) \neq 1$. By the Pontryagin duality, there exists a natural isomorphism $J : H \to \hat{\hat{H}}$ defined by

$$J(x)(\chi) = \chi(x) \ . \tag{4.82}$$

By this duality, there exists $x \in H$, such that $J(x) = \hat{\chi}$. Define $\hat{\chi}(x) := J(x)$. We have

$$\hat{\chi}(x)(\chi) = \chi(x) \ . \tag{4.83}$$

This implies that

$$\hat{\chi}\hat{G} = \{1\} \iff \forall \chi \in \hat{G},\ \chi(x) = 1 \iff \forall y \in H,\ e^{2i\sigma(x,y)} = 1 \ . \tag{4.84}$$

Continuing, using basic complex analysis, we have

$$\forall y \in H,\ e^{2i\sigma(x,y)} = 1 \iff \forall y \in H,\ \exists k \in \mathbb{Z},\ \sigma(x, y) = \pi k \tag{4.85}$$

Now, we will utilize the real linear structure of H. If $y \in H$, then $\alpha y \in H$ for all $\alpha \in \mathbb{R}$. Fix $y \in H$, then we have

$$\forall \alpha \in \mathbb{R},\ \exists k \in \mathbb{Z},\ \alpha\sigma(x, y) = \pi k \ . \tag{4.86}$$

Suppose that $\sigma(x, y) \neq 0$. Choose $\alpha = \sigma(x, y)$, then there exists a positive integer k_0 such that

$$(\sigma(x, y))^2 = \pi k_0 \ . \tag{4.87}$$

Choose $\alpha = 1$, then there exists a positive integer k_1 such that

$$\sigma(x, y) = k_1\pi \ . \tag{4.88}$$

Combining these two equations, we have

$$k_1^2\pi^2 = \pi k_0 \iff \frac{k_0}{k_1^2} = \pi \ . \tag{4.89}$$

This implies that π is rational. This is a contradiction, we must have $\sigma(x, y) = 0$. Since $y \in H$ was arbitrary, this holds for all $y \in H$. By the properties of the symplectic form σ, this implies that $x = 0$. However, if $x = 0$, then since χ is a homomorphism, we have $\chi(0) = 1$, and we thus have

$$J(x)(\chi) = \hat{\hat{\chi}}(x)(\chi) = \hat{\hat{\chi}}(0)(\chi) = \chi(0) = 1 \ . \tag{4.90}$$

This implies that $\hat{\hat{\chi}} = 1$, this is a contradiction, thus there cannot exist any $\chi \in \hat{H}$ which does not belong to $\overline{\hat{G}}$. We must have $\overline{\hat{G}} = \hat{H}$, as desired.

Applying this result, we have

$$\left|\left|\sum_{k=1}^{n} \lambda_k \hat{\pi}_{y_k,i}\right|\right| = \sup_{\chi \in \hat{G}} \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k)\right|\right| \ . \tag{4.91}$$

We need but one final observation to conclude the proof of this theorem. Let $\chi \in \hat{G}$ such that $\chi(y) = e^{2i\sigma(x,y)}$ and $\phi \in \mathcal{H}_i$ such that $||\phi|| = 1$. We have

$$\begin{aligned} W_i(x)W_i(y_k)W_i(-x)\phi &= e^{i\sigma(x,y_k)} W_i(x + y_k)W_i(-x)\phi \\ &= e^{i\sigma(x,y_k)} e^{i\sigma(x+y_k,-x)} W_i(y_k)\phi \\ &= e^{2i\sigma(x,y_k)} W_i(y_k)\phi \\ &= \chi(y_k)W_i(y_k)\phi \ . \end{aligned} \tag{4.92}$$

Applying these observations, we have

$$\begin{aligned} \sup_{\chi \in \hat{G}} \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k)\right|\right| &= \sup_{\chi \in \hat{G}} \sup_{\phi \in \mathcal{H}_i,\ ||\phi||=1} \left|\left|\sum_{k=1}^{n} \lambda_k \chi(y_k) W_i(y_k)\phi\right|\right| \\ &= \sup_{\chi \in \hat{G}} \sup_{\phi \in \mathcal{H}_i,\ ||\phi||=1} \left|\left|W_i(x) \left(\sum_{k=1}^{n} \lambda_k W_i(y_k)\right) W_i(-x)\phi\right|\right| \\ &= \left|\left|\sum_{k=1}^{n} \lambda_k W_i(y_k)\right|\right| \ . \end{aligned} \tag{4.93}$$

In the above we used the fact that if U is unitary and A is bounded, then we have $||AU|| = ||A|| = ||UA||$.

Combining all of these results, we have shown that

$$\left|\left|\sum_{k=1}^{n} \lambda_k \pi_{y_k,i}\right|\right| = \left|\left|\sum_{k=1}^{n} \lambda_k W_i(y_k)\right|\right| \ . \tag{4.94}$$

Let $\{\lambda_k\}_{k=1}^n \in \mathbb{C}$ be a finite sequence of scalars. By the earlier remarks, it remains to see that the mapping defined on the linear combinations of generators by

$$\sum_{k=1}^{n} \lambda_k \pi_{y_k,i} \mapsto \sum_{k=1}^{n} \lambda_k W_i(y_k) \tag{4.95}$$

is a linear isometry which can be extended via linearity to the closures of the linear combinations of generators. In particular, this extension is a $*$-isomorphism from Π_i to $\mathcal{B}_i$.

Finally, by unitary equivalence of the generators, the spaces Π_1 and Π_2 are $*$-isomorphic, and by combining the previous isomorphisms, we see that $\mathcal{B}_1$ and $\mathcal{B}_2$ are $*$-isomorphic. If we denote this constructed isomorphism by α, then clearly α satisfies

$$\alpha(W_1(f)) = W_2(f) \ .$$

Suppose there exists some other $*$-isomorphism γ such that

$$\gamma(W_1(f)) = W_2(f) \ .$$

It follows that α and γ agree on a dense set of generators of the corresponding CCR-algebras, and are thus the same isomorphism. □

This proof holds for any closed subalgebras of bounded operators of some Hilbert space. Every C^*-algebra is $*$-isomorphic to a closed subalgebra of bounded operators of some Hilbert space, so the proof shows that any C^*-algebras which are generated by a collection of elements satisfying the Weyl relations are mutually $*$-isomorphic.

5 Regularity of States and the Construction of Abstract Annihilation and Creation Operators

The first construction of a C^*-algebra generated by elements which satisfy the Weyl relations was done explicitly by using properties of the operators $\Phi(\cdot)$, and this construction relied heavily on the properties of these operators. On the other hand, the abstract algebraic approach has no reference to any such specific operators. Fundamentally, the framework we are trying to build is meant to be a suitable framework to solve problems of quantum statistical mechanics. As such, we are in dire need of these operators, as most other relevant operators are defined via the creation and annihilation operators.

A natural question is thus to ask whether there exists suitable analogues of the annihilation and creation operators in the abstract framework. The answer to this question is positive, albeit under some restriction in the form of regularity conditions.

The critical difference between the Weyl algebra and other algebras is that the operators of interest are not a part of the algebra itself. We can contrast this with the case of fermionic systems. The creation and annihilation operators are bounded operators, and they can thus be directly used to generate an algebra. Due to the unboundedness of the bosonic creation and annihilation operators, these operators are not a part of the Weyl algebra. We will need a more round-about approach to get a hold of these operators.

5.1 Regular States

The generators of the Weyl algebra are all unitary operators, and, if we fix $f \in \mathcal{H}$, then we can consider a semi-group given by the collection of operators $\{W(tf)\}_{t\in\mathbb{R}}$. In the case where we constructed the Weyl operators via the operators $\Phi(\cdot)$, it is obvious that this collection of operators is, in fact, strongly continuous. However, the strong continuity of the above semi-group is not guaranteed. We note that if this semi-group is strongly continuous, then, by Stone's theorem, there will exist a self-adjoint operator $\tilde{\Phi}(f)$ such that $W(tf) = e^{it\tilde{\Phi}(f)}$. This self-adjoint operator is, of course, the infinitesimal generator of the time evolution, and, as in the case of the operators $\Phi(\cdot)$, we can recuperate the annihilation and creation operators by considering linear combinations of $\tilde{\Phi}(\cdot)$.

Motivated by this discussion, we introduce the concept of a regular representation. Let $(\pi, \mathcal{H})$ be a representation of the C^*-algebra generated by a collection of elements $\{W(f) : f \in H\}$ satisfying the Weyl relations. We say that π is a regular representation if the mappings $t \mapsto \pi(W(tf))$

are strongly continuous for all $f \in H$. We say that a state ω is regular if the associated cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is regular.

As stated, we can consider the infinitesimal generators of the induced time-evolutions, and thus recuperate the $\Phi(\cdot)$-type operators. Using linear combinations of these operators, we can then define the creation and annihilation operators. As is usually the case with (possibly) unbounded operators, we must pay attention to the domains of these infinitesimal generators. The following proposition deals with the technical issues of the domains of the operators, and proves some relevant and expected properties of these operators.

Proposition 5.1. *Let $\mathcal{A}$ be the CCR algebra over the Hilbert space $\mathcal{H}$ with the symplectic form $\sigma(f,g) = -\frac{1}{2}\operatorname{Im}\langle f,g\rangle$ and ω a regular state on $\mathcal{A}$ with the associated cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$. Let $f \in \mathcal{H}$ and define $\Phi_\omega(f)$ to be the infinitesimal generator of the strongly continuous unitary semi-group $\{W(tf)\}_{t\in\mathbb{R}}$.*

It follows that for any finite dimensional subspace M of $\mathcal{H}$ the operators $\{\Phi_\omega(f), \Phi_\omega(if),\ f \in M\}$ have a common dense set of analytic vectors. Define the domains

$$D(a_\omega(f)) = D(\Phi_\omega(f)) \cap D(\Phi_\omega(if)) = D(a^*_\omega(f)) \ . \tag{5.1}$$

It follows that the operators $a_\omega(f)$ and $a^*_\omega(f)$, defined by

$$a_\omega(f) = \frac{\Phi_\omega(f) + i\Phi_\omega(if)}{\sqrt{2}},\ a^*_\omega(f) = \frac{\Phi_\omega(f) - i\Phi_\omega(if)}{\sqrt{2}}, \tag{5.2}$$

are densely defined, closed, $(a_\omega(f))^* = a^*_\omega(f)$, and, for all $\psi \in D(a_\omega(f))$, we have

$$||\Phi_\omega(f)\psi||^2 + ||\Phi_\omega(if)\psi||^2 = 2||a_\omega(f)\psi||^2 + ||f||^2||\psi||^2 \ . \tag{5.3}$$

Proof. The idea and motivation of this proof comes from the theory of semi-groups. We are going to construct the operators using a Bochner integral with a Gaussian weight. The main tool of this proof is integration by parts and some estimates for exponentials and Hermite polynomials. In fact, most of this proof is going to be spent proving technical details and estimates.

Let M be a finite dimensional subspace of $\mathcal{H}$ with dimension m. Let $\{f_j\}_{j=1}^m$ be an orthonormal basis of M and $\mathbf{f} = (f_1, ..., f_m)$. Define an operator R_n on $\mathcal{H}_\omega$ by

$$R_n = \left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right) \ . \tag{5.4}$$

In the above operator we are using the inner product brackets are meant to be understood as

$$\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle := \sum_{j=1}^m (s_j+it_j)f_j \in \mathcal{H} \ . \tag{5.5}$$

Of course, in this case we have the inner product of a vector in $\mathbb{R}^m + i\mathbb{R}^m$ and a vector in $\mathcal{H}^m$, but we should interpret the vector in $\mathcal{H}^m$ as a vector of $\mathbb{C}^m$. This operator is well-defined since it is basis independent. To see this, let $\{g_j\}_{j=1}^m$ be another orthonormal basis of M and let $\mathbf{g}$ be the corresponding row vector. The change of basis between $\mathbf{g}$ and $\mathbf{f}$ can be represented by an unitary matrix $\mathcal{M}$. Explicitly, we have $\mathbf{g} = \mathcal{M}\mathbf{f}$. By unitarity, $\mathcal{M}$ is a surjective inner product preserving operator with determinant 1. Using these observations, we have

$$\begin{aligned}
&\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right) \\
&=\int_{\mathcal{M}\mathbb{R}^m\times\mathcal{M}\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathcal{M}\mathbf{s}|^2+|\mathcal{M}\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathcal{M}\mathbf{s}+i\mathcal{M}\mathbf{t},\mathbf{g}\rangle\right)\right) \\
&=\left|\det\left(\mathcal{M}^T\right)\right|^2\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{g}\rangle\right)\right) \\
&=\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{g}\rangle\right)\right)\ .
\end{aligned} \tag{5.6}$$

Let $\psi\in\mathcal{H}_\omega$. Define $\psi_n=R_n\psi$. We will show that $||\psi_n-\psi||\to 0$. First, note that

$$\left(\frac{n}{\pi}\right)^m\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}=1\ . \tag{5.7}$$

This implies that

$$\psi_n-\psi=\left(\frac{n}{\pi}\right)^m\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\left[\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)-\mathbb{1}\right]\psi\ . \tag{5.8}$$

Let $\varepsilon>0$. By strong continuity of the regular representation, there exists $\delta>0$ such that for $(\mathbf{s},\mathbf{t})\in B(0,\delta)$, we have

$$||\left[\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)-\mathbb{1}\right]\psi||\le\varepsilon\ . \tag{5.9}$$

In the previous remark the existence of such a $\delta>0$ follows from noticing that

$$\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)=\prod_{j=1}^m\pi_\omega(W(s_jf_j))\prod_{j=1}^m\pi_\omega(W(it_jf_j)) \tag{5.10}$$

and using strong continuity separately on each operator, then choosing the smallest radius.

We have

$$\begin{aligned}
||\psi_n-\psi||&\le\left(\frac{n}{\pi}\right)^m\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}||\left[\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)-\mathbb{1}\right]\psi|| \\
&\le\varepsilon+2||\psi||\left(\frac{n}{\pi}\right)^m\int_{(\mathbb{R}^m\times\mathbb{R}^m)\setminus B(0,\delta)} d\mathbf{s}d\mathbf{t}\ e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)} \\
&=\varepsilon+2||\psi||\left(\frac{n}{\pi}\right)^m C(2m)\int_\delta^\infty dr\ r^{2m-1}e^{-nr^2} \\
&=\varepsilon+2||\psi||\left(\frac{1}{\pi}\right)^m C(2m)\int_{\sqrt{n}\delta}^\infty dr\ r^{2m-1}e^{-r^2}\ . \\
&\to\varepsilon,\ n\to\infty\ .
\end{aligned}$$

In the above calculation, the first two inequalities follow from splitting the integration domain and utilizing strong continuity, the final lines consist of doing a change of variables of the form $\sqrt{n}r\mapsto r$ and taking the limit. Since $\varepsilon>0$ was arbitrary, we have $||\psi_n-\psi||\to 0$ as desired.

Next, we will show that ψ_n are, in fact, analytic vectors of $\Phi_\omega(\cdot)$. First, we note that

$$W(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle)=W\left(\sum_{j=1}^m(s_j+it_j)f_j\right)=\prod_{j=1}^m W((s_j+it_j)f_j)\ . \tag{5.11}$$

This follows since the vectors f_j are orthogonal. Next, let $h \neq 0$, we compute

$$W((s_j + h + it_j)f_j) = e^{-\frac{iht_j}{2}} W(hf_j) W((s_j + it_j)f_j) \ . \tag{5.12}$$

For $k \in \{1, ..., m\}$, we define $\mathbf{h}_k \in \mathbb{R}^m$ by $(\mathbf{h}_k)_k = h$, and the components are 0 otherwise. By the above computation, we have

$$\begin{aligned} W(\langle \mathbf{s} + \mathbf{h}_k + i\mathbf{t}, \mathbf{f}\rangle) &= e^{-\frac{iht_k}{2}} W(hf_k) W(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle) \\ &= e^{-\frac{i\langle \mathbf{h_k}, \mathbf{t}\rangle}{2}} W(\langle \mathbf{h}_k, \mathbf{f}\rangle) W(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle) \ . \end{aligned} \tag{5.13}$$

In the calculation of this equation, we again used the orthogonality of f_j to commute $W(hf_k)$ through to the beginning of the product. Now, we compute

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \pi_\omega \left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle \right)\right) \tag{5.14}$$

$$\begin{aligned} &= \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ e^{-n(|\mathbf{s} + \mathbf{h}_k|^2 + |\mathbf{t}|^2)} \pi_\omega \left(W\left(\langle \mathbf{s} + \mathbf{h}_k + i\mathbf{t}, \mathbf{f}\rangle \right)\right) \\ &= \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ e^{-n(|\mathbf{s} + \mathbf{h}_k|^2 + |\mathbf{t}|^2)} e^{-\frac{i\langle \mathbf{h_k}, \mathbf{t}\rangle}{2}} \pi_\omega \left(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)\right) \pi_\omega \left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle \right)\right) \\ &. \end{aligned} \tag{5.15}$$

This computation shows that

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ \frac{e^{-n(|\mathbf{s} + \mathbf{h}_k|^2 + |\mathbf{t}|^2)} e^{-\frac{i\langle \mathbf{h_k}, \mathbf{t}\rangle}{2}} \pi_\omega \left(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)\right) - e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \mathbb{1}}{|\mathbf{h}_k|} \pi_\omega \left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle \right)\right) = 0 \ . \tag{5.16}$$

In order to save space, we define

$$w(\mathbf{s}, \mathbf{t}) = e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \ . \tag{5.17}$$

We have

$$\begin{aligned} &\frac{e^{-n(|\mathbf{s} + \mathbf{h}_k|^2 + |\mathbf{t}|^2)} e^{-\frac{i\langle \mathbf{h_k}, \mathbf{t}\rangle}{2}} \pi_\omega \left(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)\right) - e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \mathbb{1}}{|\mathbf{h}_k|} \\ &= \frac{e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} w(\mathbf{s} + \mathbf{h}_k, \mathbf{t}) \pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - w(\mathbf{s}, \mathbf{t}) \mathbb{1}}{|\mathbf{h}_k|} \ . \end{aligned} \tag{5.18}$$

Telescoping, we have

$$\begin{aligned} \frac{e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} w(\mathbf{s} + \mathbf{h}_k, \mathbf{t}) \pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - w(\mathbf{s}, \mathbf{t}) \mathbb{1}}{|\mathbf{h}_k|} &= e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} \frac{w(\mathbf{s} + \mathbf{h_k}, \mathbf{t}) - w(\mathbf{s}, \mathbf{t})}{|\mathbf{h}_k|} \pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) \\ &\quad + e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} w(\mathbf{s}, \mathbf{t}) \frac{\pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - \mathbb{1}}{|\mathbf{h}_k|} \ . \end{aligned} \tag{5.19}$$

Applying these computations, we have

$$\begin{aligned} &\frac{\pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - \mathbb{1}}{|\mathbf{h}_k|} \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} w(\mathbf{s}, \mathbf{t}) \pi_\omega \left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle \right)\right) \\ &= -\pi_\omega (W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ e^{-\frac{i\langle \mathbf{h}_k, t\rangle}{2}} \frac{w(\mathbf{s} + \mathbf{h_k}, \mathbf{t}) - w(\mathbf{s}, \mathbf{t})}{|\mathbf{h}_k|} \pi_\omega \left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f}\rangle \right)\right) \ . \end{aligned} \tag{5.20}$$

Applying this equation, we have

$$\frac{\pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - \mathbb{1}}{|\mathbf{h}_k|}\psi_n \tag{5.21}$$
$$= -\pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle))\left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi\ .$$

Now, writing out the vector $\mathbf{h}_k$, we can identify the left side

$$\frac{\pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - \mathbb{1}}{|\mathbf{h}_k|}\psi_n = \frac{\pi_\omega(W(hf_k)) - \mathbb{1}}{h}\psi_n\ . \tag{5.22}$$

The right-hand side would converge to the strong derivative of the mapping $t \mapsto \pi_\omega(W(tf_k))$. Since ω was a regular state, if the above norm limit exists as $h \to 0$, we have $\psi_n \in D(\Phi_\omega(f_k))$.

First, notice that we have the pointwise limit

$$\lim_{|\mathbf{h}_k|\to 0} e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|} = -2ns_k w(\mathbf{s},\mathbf{t})\ . \tag{5.23}$$

The limiting function is clearly integrable since

$$|-2ns_k w(\mathbf{s},\mathbf{t})| \le 2n|\mathbf{s}||w(\mathbf{s},\mathbf{t})|\ , \tag{5.24}$$

and the exponential of $|\mathbf{s}|^2$ will always converge faster than any polynomial of $|\mathbf{s}|$. Second, for $|\mathbf{h}_k| \le 1$, we have the bound

$$\left|\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\right| = w(\mathbf{s},\mathbf{t})\left|\frac{e^{-n(2|\mathbf{h}_k|s_k+|\mathbf{h}_k|^2)}-1}{|\mathbf{h}_k|}\right| \le w(\mathbf{s},\mathbf{t})2n(|\mathbf{s}|+1)e^{n((2|\mathbf{s}|+1))}\ . \tag{5.25}$$

The mapping $(\mathbf{s},\mathbf{t}) \mapsto w(\mathbf{s},\mathbf{t})2n(|\mathbf{s}|+1)e^{n((2|\mathbf{s}|+1))}$ is integrable, and, because $W(\cdot)$ is unitary π_ω is a $*$-homomorphism, we have

$$\left|\left|e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi\right|\right| = \left|\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\right|||\psi|| \tag{5.26}$$
$$\le w(\mathbf{s},\mathbf{t})2n(|\mathbf{s}|+1)e^{n((2|\mathbf{s}|+1))}||\psi||\ . \tag{5.27}$$

By the dominated convergence theorem for Bochner integrals, we have

$$\lim_{|\mathbf{h}_k|\to 0}\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi \tag{5.28}$$
$$= -2n\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; s_k w(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi\ .$$

Define

$$\Delta\pi_\omega = \pi_\omega(W(\langle \mathbf{h}_k, f\rangle)) - \mathbb{1},\ \Delta w(\mathbf{s},\mathbf{t}) = w(\mathbf{s}+\mathbf{h}_k,\mathbf{t}) - w(\mathbf{s},\mathbf{t}),\ w'(\mathbf{s},\mathbf{t}) = -2ns_k w(\mathbf{s},\mathbf{t})\ . \tag{5.29}$$

In this notation, we have

$$\pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle))\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi \tag{5.30}$$
$$= \pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle))\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}} w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi$$
$$+ \Delta\pi_\omega\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\left[\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|} - w'(\mathbf{s},\mathbf{t})\right]\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi$$
$$- \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}} w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi$$
$$+ \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\; e^{-\frac{i\langle \mathbf{h}_k,\mathbf{t}\rangle}{2}}\,\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t}) - w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t}, \mathbf{f}\rangle\right)\right)\psi\ .$$

We are thus left with 4 terms to evaluate. By strong continuity, we know that

$$\lim_{|\mathbf{h}_k|\to 0} \pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-\frac{i\langle \mathbf{h}_k, \mathbf{t}\rangle}{2}} w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi \tag{5.31}$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\ .$$

By a trivial application of dominated convergence, we have

$$\lim_{|\mathbf{h}_k|\to 0} \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-\frac{i\langle \mathbf{h}_k, \mathbf{t}\rangle}{2}} w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi \tag{5.32}$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\ . \tag{5.33}$$

The first and third terms thus cancel each other out.

For the second term, we have

$$\left|\left|\Delta\pi_\omega \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-\frac{i\langle \mathbf{h}_k, \mathbf{t}\rangle}{2}} \left[\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t})-w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|} - w'(\mathbf{s},\mathbf{t})\right]\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\right|\right| \tag{5.34}$$
$$\leq 2\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ \left|\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t})-w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|} - w'(\mathbf{s},\mathbf{t})\right| ||\psi||\ . \tag{5.35}$$

The pointwise limit of the integrand as $|\mathbf{h}_k| \to 0$ is 0, and, by dominated convergence using the prior bounds, we have

$$\lim_{|\mathbf{h}_k|\to 0}\left|\left|\Delta\pi_\omega \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-\frac{i\langle \mathbf{h}_k, \mathbf{t}\rangle}{2}} \left[\frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t})-w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|} - w'(\mathbf{s},\mathbf{t})\right]\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\right|\right| \tag{5.36}$$
$$= 0\ .$$

We have already shown what the fourth term converges to. Compiling these results, we have

$$\lim_{|\mathbf{h}_k|\to 0} \pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-\frac{i\langle \mathbf{h}_k, \mathbf{t}\rangle}{2}} \frac{w(\mathbf{s}+\mathbf{h_k},\mathbf{t})-w(\mathbf{s},\mathbf{t})}{|\mathbf{h}_k|}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi \tag{5.37}$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\ . \tag{5.38}$$

Which shows that

$$\lim_{|\mathbf{h}_k|\to 0} \frac{\pi_\omega(W(\langle \mathbf{h}_k, \mathbf{f}\rangle)) - \mathbb{1}}{|\mathbf{h}_k|}\psi_n = -\left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ w'(\mathbf{s},\mathbf{t})\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\ . \tag{5.39}$$

By earlier remarks, we have actually shown that

$$\Phi_\omega(f_k)\psi_n = -i\left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ 2ns_k e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\pi_\omega\left(W\left(\langle \mathbf{s}+i\mathbf{t},\mathbf{f}\rangle\right)\right)\psi\ . \tag{5.40}$$

This holds for any $k \in \{1,..,m\}$, and thus $\psi_n \in D(\Phi_\omega(f_k))$ for any $k \in \{1,...,m\}$. If $g \in M$, then there exists scalars $\{\lambda_k\}_{k=1}^m$ such that

$$g = \sum_{k=1}^m \lambda_k f_k\ . \tag{5.41}$$

For $t \in \mathbb{R}$, by orthogonality, we have

$$W(tg) = W\left(\sum_{k=1}^m t\lambda_k f_k\right) = \prod_{k=1}^m W(t\lambda_k f_k)\ . \tag{5.42}$$

Let $h \neq 0$, and for $k \in \{1, ..., m\}$ define $\mathbf{h}_k$ as before. Using orthogonality, and the fact that π_ω is a homomorphism, we have

$$\frac{\pi_\omega(W(hg)) - \mathbb{1}}{h}\psi_n = \sum_{k=1}^{m-1} \prod_{j=k+1}^{m} \pi_\omega(W(h\lambda_j f_j)) \frac{W(\langle \lambda_k \mathbf{h}_k, f_k \rangle) - \mathbb{1}}{|\mathbf{h}_k|}\psi_n + \frac{W(\langle \lambda_m \mathbf{h}_m, f_m \rangle) - \mathbb{1}}{|\mathbf{h}_m|}\psi_n \ . \tag{5.43}$$

Now, observe that

$$\begin{aligned} &\left|\left| \prod_{j=k+1}^{m} \pi_\omega(W(h\lambda_j f_j)) \left(\frac{W(\langle \lambda_k \mathbf{h}_k, f_k \rangle) - \mathbb{1}}{|\mathbf{h}_k|} - i\lambda_k \Phi_\omega(f_k) \right) \psi_n \right|\right| \\ &= \left|\left| \left(\frac{W(\langle \lambda_k \mathbf{h}_k, f_k \rangle) - \mathbb{1}}{|\mathbf{h}_k|} - i\lambda_k \Phi_\omega(f_k) \right) \psi_n \right|\right| \to 0, \ h \to 0 \ , \end{aligned} \tag{5.44}$$

and, using orthogonality, we have

$$\begin{aligned} &\left|\left| \prod_{j=k+1}^{m} \pi_\omega(W(h\lambda_j f_j)) i\lambda_k \Phi_\omega(f_k)\psi_n - i\lambda_k \Phi_\omega(f_k)\psi_n \right|\right| \\ &\leq \sum_{j=k+1}^{m} \left|\left| \prod_{l=j+1}^{m} \pi_\omega(W(h\lambda_l f_l))(\pi_\omega W(h\lambda_j f_j)) - \mathbb{1}) i\lambda_k \Phi_\omega(f_k)\psi_n \right|\right| \\ &\leq \sum_{j=k+1}^{m} ||(\pi_\omega W(h\lambda_j f_j)) - \mathbb{1}) i\lambda_k \Phi_\omega(f_k)\psi_n|| \to 0, \ h \to 0 \ . \end{aligned} \tag{5.45}$$

Combining the two inequalities above, and telescoping appropriately, we have

$$\lim_{h \to 0} \frac{\pi_\omega(W(hg)) - \mathbb{1}}{h}\psi_n = \sum_{k=1}^{m} i\lambda_k \Phi_\omega(f_k)\psi_n \ . \tag{5.46}$$

This lengthy computation shows that for any $g \in M$, we have $\psi_n \in D(\Phi_\omega(g))$, and

$$\Phi_\omega(g)\psi_n = \sum_{k=1}^{m} \lambda_k \Phi_\omega(f_k)\psi_n = -i\left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s}d\mathbf{t} \ 2n \sum_{k=1}^{n} \lambda_k s_k e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \pi_\omega\left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f} \rangle\right)\right)\psi \ . \tag{5.47}$$

Next, let $p \in \mathbb{N}$. We will sketch of how to compute $(\Phi_\omega(\cdot))^p\psi_n$. First, we remark that the "method" that was used to compute $\Phi(f_k)\psi_n$ was basically a glorified version of integration by parts. Let $k, l \in \{1, ..., m\}$. Iterating the same proof, it is not difficult to see that $\Phi_\omega(f_k)$ and $\Phi_\omega(f_l)$ commute. Because of this commutation, we have

$$(\Phi_\omega(g))^p\psi_n = \sum_{k_1 + k_2 ... + k_m = p} \binom{p}{k_1, \ k_2, \ ..., \ k_m} \prod_{l=1}^{m} (\lambda_l \Phi_\omega(f_l))^{k_l}\psi_n \ . \tag{5.48}$$

To make the next step clearer, we note that

$$\Phi_\omega(f_k)\psi_n = i\left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s}d\mathbf{t} \ \frac{\partial}{\partial s_k}\left(e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)}\right) \pi_\omega\left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f} \rangle\right)\right)\psi \ . \tag{5.49}$$

In general, we have

$$\prod_{l=1}^{m} (\lambda_l \Phi_\omega(f_l))^{k_l}\psi_n = (-1)^{p+1} i^p \left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s}d\mathbf{t} \ \prod_{l=1}^{m} \lambda_l^{k_l} \frac{\partial^{k_l}}{\partial s_l^{k_l}} \left(e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)}\right) \pi_\omega\left(W\left(\langle \mathbf{s} + i\mathbf{t}, \mathbf{f} \rangle\right)\right)\psi \ . \tag{5.50}$$

We have

$$\left|\left|\prod_{l=1}^{m}(\lambda_l\Phi_\omega(f_l))^{k_l}\psi_n\right|\right| \le ||\psi|| \left(\frac{n}{\pi}\right)^m \int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ \prod_{l=1}^{m}|\lambda_l|^{k_l}\left|\frac{\partial^{k_l}}{\partial s_l^{k_l}}\left(e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\right)\right| , \tag{5.51}$$

and, subsequently, we have

$$\begin{aligned}&\sum_{p=0}^{P}\frac{t^p}{p!}||(\Phi_\omega(g))^p\psi_n|| \\ &\le ||\psi||\sum_{p=0}^{P}\frac{t^p}{p!}\sum_{k_1+k_2...+k_m=p}\binom{p}{k_1,\ k_2,\ ...,\ k_m}\left(\frac{n}{\pi}\right)^m\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ \prod_{l=1}^{m}|\lambda_l|^{k_l}\left|\frac{\partial^{k_l}}{\partial s_l^{k_l}}\left(e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\right)\right| .\end{aligned} \tag{5.52}$$

If we can show that the sum on the right converges as $P\to\infty$, then we have shown that ψ_n are analytic vectors of $\Phi_\omega(g)$, and since $\psi_n\to\psi\in\mathcal{H}_\omega$, it follows that $\Phi_\omega(g)$ has a dense set of analytic vectors for every g.

First, note that

$$\frac{\partial^{k_l}}{\partial s_l^{k_l}}\left(e^{-n|\mathbf{s}|^2}\right) = \prod_{l=1}^{m}\frac{\partial^{k_l}}{\partial s_l^{k_l}}e^{-ns_l^2} .$$

Using the above to separate the integrand appropriately, we have

$$\begin{aligned}&\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ \prod_{l=1}^{m}|\lambda_l|^{k_l}\left|\frac{\partial^{k_l}}{\partial s_l^{k_l}}\left(e^{-n(|\mathbf{s}|^2+|\mathbf{t}|^2)}\right)\right| \\ &\le \left(\max_{l\in\{1,...,m\}}|\lambda_l|\right)^p\int_{\mathbb{R}^m\times\mathbb{R}^m} d\mathbf{s}d\mathbf{t}\ e^{-n|\mathbf{t}|^2}\prod_{l=1}^{m}\left|\frac{\partial^{k_l}}{\partial s_l^{k_l}}e^{-ns_l^2}\right| \\ &= \left(\max_{l\in\{1,...,m\}}|\lambda_l|\right)^p\left(\frac{\pi}{n}\right)^{\frac{m}{2}}\int_{\mathbb{R}^m} d\mathbf{s}\ \prod_{l=1}^{m}\left|\frac{\partial^{k_l}}{\partial s_l^{k_l}}e^{-ns_l^2}\right| \\ &= \left(\max_{l\in\{1,...,m\}}|\lambda_l|\right)^p\left(\frac{\pi}{n}\right)^{\frac{m}{2}}\prod_{l=1}^{m}\int_{\mathbb{R}} ds\ \left|\frac{\partial^{k_l}}{\partial s^{k_l}}e^{-ns^2}\right| .\end{aligned} \tag{5.53}$$

In the above, to get the exponent p, we used the fact that we are summing over indices which satisfy $k_1+k_2+...+k_m=p$. We see that an appropriate bound can be found by bounding

$$\int_{\mathbb{R}} ds\ \left|\frac{\partial^{k_l}}{\partial s^{k_l}}e^{-ns^2}\right| . \tag{5.54}$$

Let $k\in\mathbb{N}$ and define $x(s)=\sqrt{n}s$. We have

$$\frac{\partial^k}{\partial s^k}\left(e^{-ns^2}\right) = n^{\frac{k}{2}}\frac{\partial^k}{\partial x^k}\left(e^{-x^2}\right) . \tag{5.55}$$

Recall that the derivative is related to the k-th Hermite polynomial H_k by

$$\frac{\partial^k}{\partial x^k}\left(e^{-x^2}\right) = (-1)^k e^{-x^2}H_k(x) . \tag{5.56}$$

We thus have

$$\left|\frac{\partial^k}{\partial s^k}\left(e^{-ns^2}\right)\right| \le n^{\frac{k}{2}}|H_k(\sqrt{n}s)|e^{-ns^2} . \tag{5.57}$$

It has been shown by Jack Indritz in [7] that

$$|H_k(\sqrt{n}s)| \leq (2^k k!)^{\frac{1}{2}} e^{\frac{ns^2}{2}} \ . \tag{5.58}$$

We thus have

$$\left| \frac{\partial^k}{\partial s^k} \left(e^{-ns^2} \right) \right| \leq n^{\frac{k}{2}} (2^k k!)^{\frac{1}{2}} e^{-\frac{ns^2}{2}} \ . \tag{5.59}$$

Applying this bound, we have

$$\int_{\mathbb{R}} ds \ \left| \frac{\partial^{k_l}}{\partial s^{k_l}} e^{-ns^2} \right| \leq n^{\frac{k}{2}} (2^k k!)^{\frac{1}{2}} \int_{\mathbb{R}} ds \ e^{-\frac{ns^2}{2}} = \frac{(2\pi)^{\frac{1}{2}} n^{\frac{k}{2}} (2^k k!)^{\frac{1}{2}}}{n^{\frac{1}{2}}} \ . \tag{5.60}$$

Applying this bound, we have

$$\left(\max_{l \in \{1,\ldots,m\}} |\lambda_l| \right)^p \left(\frac{\pi}{n} \right)^{\frac{m}{2}} \prod_{l=1}^{m} \int_{\mathbb{R}} ds \ \left| \frac{\partial^{k_l}}{\partial s^{k_l}} e^{-ns^2} \right| \tag{5.61}$$

$$\leq 2^{\frac{m}{2}} \left(\max_{l \in \{1,\ldots,m\}} |\lambda_l| \right)^p \left(\frac{\pi}{n} \right)^m n^{\frac{p}{2}} 2^{\frac{p}{2}} \left(\prod_{l=1}^{m} k_l! \right)^{\frac{1}{2}} \ . \tag{5.62}$$

Gathering these results, we have

$$\left(\frac{n}{\pi} \right)^m \int_{\mathbb{R}^m \times \mathbb{R}^m} d\mathbf{s} d\mathbf{t} \ \prod_{l=1}^{m} |\lambda_l|^{k_l} \left| \frac{\partial^{k_l}}{\partial s_l^{k_l}} \left(e^{-n(|\mathbf{s}|^2 + |\mathbf{t}|^2)} \right) \right| \leq 2^{\frac{m}{2}} \left(\max_{l \in \{1,\ldots,m\}} |\lambda_l| \right)^p n^{\frac{p}{2}} 2^{\frac{p}{2}} \left(\prod_{l=1}^{m} k_l! \right)^{\frac{1}{2}} \ . \tag{5.63}$$

Applying this bound to eq. (5.52), we have

$$\sum_{p=0}^{P} \frac{t^p}{p!} ||(\Phi_\omega(g))^p \psi_n|| \tag{5.64}$$

$$\leq 2^{\frac{m}{2}} ||\psi|| \sum_{p=0}^{P} \frac{\left(t \left(\max_{l \in \{1,\ldots,m\}} |\lambda_l| \right) n^{\frac{1}{2}} 2^{\frac{1}{2}} \right)^p}{p!} \sum_{k_1 + k_2 \ldots + k_m = p} \binom{p}{k_1, \ k_2, \ \ldots, \ k_m} \left(\prod_{l=1}^{m} k_l! \right)^{\frac{1}{2}} \ .$$

Applying Cauchy-Schwartz, we have

$$\begin{aligned} \sum_{k_1 + k_2 \ldots + k_m = p} \binom{p}{k_1, \ k_2, \ \ldots, \ k_m} \left(\prod_{l=1}^{m} k_l! \right)^{\frac{1}{2}} &= (p!)^{\frac{1}{2}} \sum_{k_1 + \ldots + k_m = p} \frac{(p!)^{\frac{1}{2}}}{\left(\prod_{l=1}^{m} k_l! \right)^{\frac{1}{2}}} \\ &\leq (p!)^{\frac{1}{2}} \left(\sum_{k_1 + \ldots + k_m = p} \binom{p}{k_1, \ k_2, \ \ldots, \ k_m} \sum_{k_1 + \ldots + k_m = p} 1 \right)^{\frac{1}{2}} \\ &= (p!)^{\frac{1}{2}} m^{\frac{p}{2}} \binom{p+m-1}{m-1}^{\frac{1}{2}} \\ &= \frac{(p!)^{\frac{1}{2}} m^{\frac{p}{2}} ((p+m-1)!)^{\frac{1}{2}}}{((m-1)!)^{\frac{1}{2}} (p!)^{\frac{1}{2}}} \ . \end{aligned} \tag{5.65}$$

In the above, we used two identities. The first one is simply

$$\sum_{k_1 + \ldots + k_m = p} \binom{p}{k_1, \ k_2, \ \ldots, \ k_m} = \left(\sum_{k=1}^{m} 1 \right)^p = m^p, \tag{5.66}$$

and the second one is

$$\sum_{k_1+\ldots+k_m=p} 1 = \binom{p+m-1}{m-1} \ . \tag{5.67}$$

The second identity is the answer to the combinatorial question : Given m bars, how many ways are there to distribute p stars in between the m bars? It is thus a matter of applying the stars and bars method which gives us our identity.

We thus have

$$\begin{aligned} &2^{\frac{m}{2}}||\psi||\sum_{p=0}^{P}\frac{\left(t\left(\max_{l\in\{1,\ldots,m\}}|\lambda_l|\right)n^{\frac{1}{2}}2^{\frac{1}{2}}\right)^p}{p!}\sum_{k_1+k_2\ldots+k_m=p}\binom{p}{k_1,\ k_2,\ \ldots,\ k_m}\left(\prod_{l=1}^{m}k_l!\right)^{\frac{1}{2}} \\ &\leq \frac{2^{\frac{m}{2}}||\psi||}{((m-1)!)^{\frac{1}{2}}}\sum_{p=0}^{P}\left(t\left(\max_{l\in\{1,\ldots,m\}}|\lambda_l|\right)(mn)^{\frac{1}{2}}2^{\frac{1}{2}}\right)^p\frac{((p+m-1)!)^{\frac{1}{2}}}{p!} \ . \end{aligned} \tag{5.68}$$

Finally, we have

$$\frac{\frac{((p+m)!)^{\frac{1}{2}}}{(p+1)!}}{\frac{((p+m-1)!)^{\frac{1}{2}}}{p!}} = \frac{(p+m)^{\frac{1}{2}}}{p+1} \to 0,\ p\to\infty \ . \tag{5.69}$$

This implies that the series

$$\sum_{p=0}^{P}\left(t\left(\max_{l\in\{1,\ldots,m\}}|\lambda_l|\right)(mn)^{\frac{1}{2}}2^{\frac{1}{2}}\right)^p\frac{((p+m-1)!)^{\frac{1}{2}}}{p!} \tag{5.70}$$

converges as $P\to\infty$. By earlier observations, this implies that

$$\sum_{p=0}^{\infty}\frac{t^p}{p!}||(\Phi_\omega(g))^p\psi_n|| < \infty \ . \tag{5.71}$$

To summarize what we have shown. For any $\psi\in\mathcal{H}_\omega$, there exists a collection of vectors $\{\psi_n\}_{n\in\mathbb{N}}$ such that each ψ_n is an analytic vector of $\Phi_\omega(g)$, for all $g\in M$, and $\psi_n\to\psi$. To put it more succinctly, for every $g\in M$, the operators $\Phi_\omega(g)$ all have and share a dense set of analytic vectors.

By definitions of the domains of $a_\omega(g)$ and $a^*_\omega(g)$, we immediately see that these are densely defined operators. Next, let $\phi\in D(a^*_\omega(g))$ and $\eta\in D(a_\omega(g))$. A simple calculation shows that

$$\langle\eta, a^*_\omega(g)\phi\rangle = \langle a_\omega(g)\eta,\phi\rangle \ . \tag{5.72}$$

The adjoint of $a_\omega(g)$ is the unique operator which satisfies the above equation. Since $a^*_\omega(g)$ also satisfies this equation, we see that $(a_\omega(g))^*$ is an extension of $a^*_\omega(g)$. Since $a^*_\omega(g)$ is a densely defined operator, it follows that $(a_\omega(g))^*$ is also a densely defined operator, and the operator $a_\omega(g)$ thus has a densely defined adjoint which implies that $a_\omega(g)$ is closable. Applying the same logic to $a^*_\omega(g)$, we see that it too is closable.

Let $\phi\in D(a_\omega(g))$. A simple calculation shows that

$$||\Phi_\omega(g)\phi||^2 + ||\Phi_\omega(ig)\phi||^2 = ||a_\omega(g)\phi||^2 + ||a^*_\omega(g)\phi||^2 \ . \tag{5.73}$$

Going further, using the unitarity of the given operators, we have

$$\begin{aligned} \langle\Phi_\omega(g)\phi,\Phi_\omega(ig)\phi\rangle &= \lim_{h\to 0}\left\langle\frac{\pi_\omega(W(hg))-\mathbb{1}}{h}\phi,\frac{\pi_\omega(W(hig))-\mathbb{1}}{h}\phi\right\rangle \\ &= \lim_{h\to 0}\left\langle\frac{\pi_\omega(W(-hig))-\mathbb{1}}{h}\frac{\pi_\omega(W(hg))-\mathbb{1}}{h}\phi,\phi\right\rangle \ . \end{aligned} \tag{5.74}$$

Applying the commutation relations, and telescoping appropriately, we have

$$\frac{\pi_\omega(W(-hig)) - \mathbb{1}}{h} \frac{\pi_\omega(W(hg)) - \mathbb{1}}{h} = \frac{e^{\frac{-ih^2||g||^2}{2}} - 1}{h^2} W(-hig)W(hg) + \frac{\pi_\omega(W(hg)) - \mathbb{1}}{h} \frac{\pi_\omega(W(-hig)) - \mathbb{1}}{h} \ . \tag{5.75}$$

Going back to the original equation, we have

$$\left\langle \frac{\pi_\omega(W(-hig)) - \mathbb{1}}{h} \frac{\pi_\omega(W(hg)) - \mathbb{1}}{h} \phi, \phi \right\rangle = \frac{e^{-ih^2||g||^2} - 1}{h^2} \langle W(-hig)\phi, W(-hg)\phi \rangle + \left\langle \frac{\pi_\omega(W(-ihg)) - \mathbb{1}}{h} \phi, \frac{\pi_\omega(W(-hg)) - \mathbb{1}}{h} \phi \right\rangle \ . \tag{5.76}$$

In the above, we used the commutation relations to compute

$$\begin{aligned} W(-hig)W(hg) = e^{-i\frac{\mathrm{Im}\langle hig, hg\rangle}{2}} W(h(1-i)g) &= e^{-i\frac{h^2||g||^2}{2}} e^{-i\frac{h^2||g||^2}{2}} W(hg)W(-hig) \\ &= e^{-ih^2||g||^2} W(hg)W(-hig) \ . \end{aligned} \tag{5.77}$$

Letting $h \to 0$, we have

$$\langle \Phi_\omega(g)\phi, \Phi_\omega(ig)\phi \rangle = -i||g||^2||\phi||^2 + \langle \Phi_\omega(ig)\phi, \Phi_\omega(g)\phi \rangle \ . \tag{5.78}$$

Equipped with this equality, we have

$$\begin{aligned} ||a_\omega(g)\phi||^2 - ||a^*_\omega(g)\phi||^2 &= \left|\left| \frac{\Phi_\omega(g) + i\Phi_\omega(ig)}{\sqrt{2}} \phi \right|\right|^2 - \left|\left| \frac{\Phi_\omega(g) - i\Phi_\omega(ig)}{\sqrt{2}} \phi \right|\right|^2 \\ &= i \frac{\langle \Phi_\omega(ig)\phi, \Phi_\omega(g)\phi \rangle - \langle \Phi_\omega(g)\phi, \Phi_\omega(ig)\phi \rangle}{2} \\ &- i \frac{\langle \Phi_\omega(g)\phi, \Phi_\omega(ig)\phi \rangle - \langle \Phi_\omega(ig)\phi, \Phi_\omega(g)\phi \rangle}{2} \\ &= -||g||^2||\phi||^2 \ . \end{aligned} \tag{5.79}$$

Combining the above equation and eq. (5.73), we have

$$||\Phi_\omega(g)\phi||^2 + ||\Phi_\omega(ig)\phi||^2 = 2||a_\omega(g)\phi||^2 + ||g||^2||\phi||^2 \ . \tag{5.80}$$

We are now in a position to show that the operators $a_\omega(g)$ are closed. Let $\phi \in D(a_\omega(g))$ such that there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ such that $\phi_n \to \phi$ and $a_\omega(g)\phi_n \to \eta$ for some $\eta \in \mathcal{H}_\omega$. By the previous equation, we have

$$||\Phi_\omega(g)(\phi_n - \phi_m)||^2 + ||\Phi_\omega(ig)(\phi_n - \phi_m)||^2 = 2||a_\omega(g)(\phi_n - \phi_m)||^2 + ||g||^2||\phi_n - \phi_m||^2 \tag{5.81}$$

The operators $\Phi_\omega(\cdot)$ are self-adjoint and thus they are closed. Furthermore, we have $\phi_n - \phi_m \in D(\Phi_\omega(g)) \cap D(\Phi_\omega(ig))$, and $\phi_n - \phi_m \to 0,\ n, m \to \infty$. We have

$$||\Phi_\omega(g)(\phi_n - \phi_m)||^2 + ||\Phi_\omega(ig)(\phi_n - \phi_m)||^2 \to 0,\ n, m \to \infty \ . \tag{5.82}$$

This implies that the sequence with terms $\Phi_\omega(g)\phi_n$ is a Cauchy sequence and hence converges. Because $\Phi_\omega(\cdot)$ are closed operators and $\phi \in D(\Phi_\omega(g))$, we must have

$$\Phi_\omega(g)\phi_n \to \Phi_\omega(g)\phi,\ n \to \infty \ , \tag{5.83}$$

and the same for $\Phi_\omega(ig)$. Reusing the previous equation, we have

$$||\Phi_\omega(g)(\phi_n - \phi)||^2 + ||\Phi_\omega(ig)(\phi_n - \phi)||^2 = 2||a_\omega(g)(\phi_n - \phi)||^2 + ||g||^2||\phi_n - \phi||^2 \ . \tag{5.84}$$

Letting $n \to \infty$, we have

$$||a_\omega(g)(\phi_n - \phi)|| \to 0, \; n \to \infty \; , \tag{5.85}$$

which implies that $\eta = a_\omega(g)\phi$. The operator $a_\omega(g)$ is thus closed. The same argument shows that $a^*_\omega(g)$ is closed.

We have shown that $(a_\omega(g))^*$ is an extension of $a^*_\omega(g)$. Unfortunately, the proof that $a^*_\omega(g)$ is also an extension of $(a_\omega(g))^*$ and thus $a^*_\omega(g) = (a_\omega(g))^*$ would lead us too far astray from our main topic. The proof of this relies on the Stone-von Neumann theorem regarding the uniqueness of some finite dimensional algebras. Furthermore, one must introduce the concept of normal states and normal representation which will not play a part in the rest of this thesis. For the interested reader, it is encouraged to examine the full proof of the Stone-von Neumann theorem along with the corollary of our desired result in [1, p. 34, Corollary 5.2.15]. □

5.2 Analytic States

Regular states and representations thus give us a natural construction of the annihilation and creation operators. Next, we are interested in seeing to what extent we can utilize the creation and annihilation operators to fully specify states in the CCR-algebra over $\mathcal{H}$. Given a state ω on a C^*-algebra which is generated by some elements, by continuity, it is enough to check the value of the state ω for each multivariable polynomial with variables given by the generators of the C^*-algebra. Recall the discussion on the interplay multiplicative and linear structure in the CCR-algebra. In particular, we noted that multivariable polynomials of the generators of the CCR-algebra are, in fact, linear combinations of the generators. The states of the CCR-algebra are linear functionals, and, fortuitously, it is enough to check the value of the state on each generator in this special case. In our case, the generators of the CCR-algebra are exactly the Weyl operators, or, more specifically, their representations on the respective operator algebra. In most practical cases, the annihilation and creation operators are unbounded operators, and thus cannot belong to the CCR-algebra.

This previous point can be seen by considering a regular cyclic representation $(\pi_\omega, \mathcal{H}_\omega . \Omega_\omega)$ of the CCR-algebra $\mathcal{A}$ on $\mathcal{H}$ for a state ω on the CCR-algebra. If $A \in \mathcal{A}$, then by cyclicity, we have

$$\omega(A) = \langle \pi_\omega(A)\Omega_\omega, \Omega_\omega \rangle \; . \tag{5.86}$$

Let $f \in \mathcal{H}$. Define $\Phi_\omega(f)$ as in proposition 5.1, and, informally, we try to define

$$\omega(\Phi_\omega(f)) = \langle \Phi_\omega(f)\Omega_\omega, \Omega_\omega \rangle \; . \tag{5.87}$$

If the cyclic vector Ω_ω does not belong to the domain of $\Phi_\omega(f)$, then the operator $\Phi_\omega(f)$ is necessarily unbounded, and we cannot hope to define the value of the state.

Our ultimate goal here is to rid ourselves of the dependence on the Weyl operators when defining the values of a state ω. We can motivate the next definition and result with an informal calculation. As stated, it is enough to give the value of the state for the generators of the corresponding algebra.

Let ω be a regular state as before, for any $f \in \mathcal{H}$ and $t \in \mathbb{R}$, we have

$$\omega(W(tf)) = \langle \pi_\omega(W(tf))\Omega_\omega, \Omega_\omega \rangle \; . \tag{5.88}$$

Here, we have used the natural notation where we omit the π_ω in the input of the state. Suppose that Ω_ω is an analytic vector of $\pi_\omega(W(tf))$. Explicitly, $\Omega_\omega \in C^\infty(\Phi_\omega(f))$ and, for all $t \in \mathbb{R}$, we have

$$\sum_{n=0}^{\infty} \frac{t^n ||\Phi_\omega(f)^n \Omega_\omega||}{n!} < \infty, \tag{5.89}$$

and

$$\pi_\omega(W(tf))\Omega_\omega = \sum_{n=0}^{\infty} \frac{(it\Phi_\omega(f))^n\Omega_\omega}{n!} \ . \tag{5.90}$$

Now, applying continuity of the inner product, we have

$$\omega(W(tf)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle (i\Phi_\omega(f))^n\Omega_\omega, \Omega_\omega\rangle \ . \tag{5.91}$$

We see that values of the state are completely determined by the family of expectations $\{\langle(\Phi_\omega(f))^n\Omega_\omega, \Omega_\omega\rangle : n \in \mathbb{N},\ f \in \mathcal{H}\}$.

Conversely, suppose that $t \mapsto \omega(W(tf))$ is analytic in a neighbourhood containing the origin. This mapping is completely determined by the value of its derivatives at the origin. First, we will show that $t \mapsto \pi_\omega(W(tf))\Omega_\omega$ is weakly differentiable.

First, we will need a bound on the difference quotient of $\pi_\omega(W(hf))$. For $h \neq 0$, using the commutation relations, we compute

$$\begin{aligned} ||(\pi_\omega(W(hf)) - \mathbb{1})\Omega_\omega||^2 &= \langle(\pi_\omega(W(hf)) - \mathbb{1})\Omega_\omega, (\pi_\omega(W(hf)) - \mathbb{1})\Omega_\omega\rangle \\ &= \langle(\pi_\omega(W(-hf)) - \mathbb{1})(\pi_\omega(W(hf)) - \mathbb{1})\Omega_\omega, \Omega_\omega\rangle \\ &= 1 - \omega(W(-hf)) + 1 - \omega(W(hf)) \ . \end{aligned} \tag{5.92}$$

Because the mapping $t \mapsto \omega(W(tf))$ is analytic, we know that there exists coefficients $\{a_k\}_{k\in\mathbb{N}} \in \mathbb{C}$ such that

$$\omega(W(tf)) = \sum_{k=0}^{\infty} a_k t^k \ . \tag{5.93}$$

In particular, we see that $1 = \omega(W(0)) = a_0$. Applying this to eq. (5.92), we have

$$\begin{aligned} 1 - \omega(W(-hf)) + 1 - \omega(W(hf)) &= 1 - 1 - \sum_{k=1}^{\infty} a_k h^k + 1 - 1 + 2\sum_{k=1}^{\infty} a_k(-h)^k \\ &= -2a_2h^2 - 2h^2\sum_{k=1}^{\infty} a_{2(k+1)}h^{2k} \ . \end{aligned} \tag{5.94}$$

We thus have

$$\left|\left|\left(\frac{\pi_\omega(W(hf)) - \mathbb{1}}{h}\right)\Omega_\omega\right|\right|^2 = -2a_2 - 2\sum_{k=1}^{\infty} a_{2(k+1)}h^{2k} \ . \tag{5.95}$$

Letting $h \to 0$, we see that

$$\lim_{h\to 0}\left|\left|\left(\frac{\pi_\omega(W(hf)) - \mathbb{1}}{h}\right)\Omega_\omega\right|\right| = \sqrt{2|a_2|} < \infty \ . \tag{5.96}$$

This implies that $\Omega_\omega \in D(\Phi_\omega(f))$, as desired.

For any $t \in \mathbb{R}$, using the commutation relations, we have

$$\begin{aligned} \lim_{h\to 0}\frac{\omega(W((t+h)f)) - \omega(W(tf))}{h} &= \lim_{h\to 0}\left\langle \pi_\omega(W(tf))\frac{\pi_\omega(W(hf)) - \mathbb{1}}{h}\Omega_\omega, \Omega_\omega\right\rangle \\ &= \langle i\Phi_\omega(f)\Omega_\omega, \pi_\omega(W(-tf))\Omega_\omega\rangle \\ &= \langle \pi_\omega(W(tf))(i\Phi_\omega(f))\Omega_\omega, \Omega_\omega\rangle \ . \end{aligned}$$

Now, even though $\Phi_\omega(f) \notin \mathcal{A}$, we can extend the state ω to include $\Phi_\omega(f)$ by using the existence of the above limit. In natural notation, we define

$$\omega(\Phi_\omega(f)) := \lim_{h\to 0} \frac{1}{i} \frac{\omega(W(hf)) - \omega(\mathbb{1})}{h} = \langle \Phi_\omega(f)\Omega_\omega, \Omega_\omega \rangle \ . \tag{5.97}$$

Next, we will show that $\pi_\omega(W(f))\Omega_\omega \in D(\Phi_\omega(f))$. Indeed, using the commutation relations, we have

$$\left\langle \psi, \frac{\pi_\omega(W(hf)) - \mathbb{1}}{h} \pi_\omega(W(f))\Omega_\omega \right\rangle = \left\langle \pi_\omega(W(-f))\psi, \frac{\pi_\omega(W(hf)) - \mathbb{1}}{h} \Omega_\omega \right\rangle \ . \tag{5.98}$$

Utilizing the weak derivative of the mapping $t \mapsto \pi_\omega(W(tf))$, we have

$$\lim_{h\to 0} \left\langle \psi, \frac{\pi_\omega(W(hf)) - \mathbb{1}}{h} \pi_\omega(W(f))\Omega_\omega \right\rangle = \langle \psi, i\pi_\omega(W(f))\phi \rangle \ . \tag{5.99}$$

We see that the weak derivative of the mapping $t \mapsto \omega(W(tf)W(f))$ is $i\pi_\omega W(f)\phi$. By Stone's theorem, we have $\pi_\omega(W(f))\Omega_\omega \in D(\Phi_\omega(f))$

Next, we will give a sketch of the proof to show that $\Omega_\omega \in D((\Phi_\omega(f))^n)$. We will mainly focus on the techniques that will be needed to bring the proof to its completion. The induction is tediously long and unenlightening.

In eq. (5.96), we note that the crucial part of this proof is the bound on the term

$$\left|\left| \frac{\pi_\omega(W(hf)) - \mathbb{1}}{h} \Omega_\omega \right|\right| \ . \tag{5.100}$$

In the previous proof, we showed that the derivative of the mapping $t \mapsto \omega(W(tf))$ is given by $\omega(W(tf)(i\Phi_\omega(f)))$. We see that for the proof that $\Omega_\omega \in D((\Phi_\omega(f))^2)$ it will be necessary to find a bound for

$$\left|\left| \frac{\pi_\omega(W(hf)) - \mathbb{1}}{h} \Phi_\omega(f)\Omega_\omega \right|\right| \ . \tag{5.101}$$

As before, we compute

$$\begin{aligned} ||(\pi_\omega(W(hf)) - \mathbb{1})\Phi_\omega(f)\Omega_\omega||^2 &= \langle (\mathbb{1} - \pi_\omega(W(hf)))\Phi_\omega(f)\Omega_\omega, \Phi_\omega(f)\Omega_\omega \rangle \\ &\quad + \langle (\mathbb{1} - \pi_\omega(W(-hf)))\Phi_\omega(f)\Omega_\omega, \Phi_\omega(f)\Omega_\omega \rangle \ . \end{aligned} \tag{5.102}$$

We have

$$\begin{aligned} &\langle (\mathbb{1} - \pi_\omega(W(hf)))\Phi_\omega(f)\Omega_\omega, \Phi_\omega(f)\Omega_\omega \rangle \\ &= \lim_{w\to 0} \left\langle (\mathbb{1} - \pi_\omega(W(hf)))(i\Phi_\omega(f))\Omega_\omega, \frac{\pi_\omega(W(wf)) - \mathbb{1}}{w} \Omega_\omega \right\rangle \\ &= \lim_{w\to 0} \left\langle \frac{\pi_\omega(W(-wf)) - \pi_\omega(W((h-w)f)) - \mathbb{1} + \pi_\omega(W(hf))}{w} (i\Phi_\omega(f))\Omega_\omega, \Omega_\omega \right\rangle \ . \end{aligned} \tag{5.103}$$

Now, the mapping $t \mapsto \omega(W(tf)\Phi_\omega(f))$ is analytic, there thus exists coefficients $\{b_k\}_{k\in\mathbb{N}} \in \mathbb{C}$ such that

$$\begin{aligned} &\left\langle \frac{\pi_\omega(W(-wf)) - \pi_\omega(W((h-w)f)) - \mathbb{1} + \pi_\omega(W(hf))}{w} (i\Phi_\omega(f))\Omega_\omega, \Omega_\omega \right\rangle \\ &= \frac{\sum_{k=0}^\infty b_k((-w)^k - (h-w)^k + h^k) - \omega(\Phi_\omega(f))}{w} \ . \end{aligned} \tag{5.104}$$

As before, we immediately see that $b_0 = \omega(\Phi_\omega(f))$, and, going further, we compute

$$\begin{aligned}
&\frac{\sum_{k=0}^{\infty} b_k((-w)^k - (h-w)^k + h^k) - \omega(\Phi_\omega(f))}{w} \\
&= 2hb_2 + \sum_{k=3}^{\infty} b_k(-1)^k w^{k-1} - \sum_{k=3}^{\infty} b_k \frac{(h+(-w))^k - h^k}{(-w)} \ .
\end{aligned} \tag{5.105}$$

Letting $w \to 0$, we have

$$\lim_{w\to 0}\left(2hb_2 + \sum_{k=3}^{\infty} b_k(-1)^k w^{k-1} - \sum_{k=3}^{\infty} b_k \frac{(h+(-w))^k - h^k}{(-w)}\right) = 2hb_2 - \sum_{k=3}^{\infty} b_k k h^{k-1} \ . \tag{5.106}$$

To summarize, we have shown that

$$\langle(\mathbb{1} - \pi_\omega(W(hf)))\Phi_\omega(f)\Omega_\omega, \Phi_\omega(f)\Omega_\omega\rangle = 2hb_2 - \sum_{k=3}^{\infty} b_k k h^{k-1} \ . \tag{5.107}$$

Applying this equality to eq. (5.102), we have

$$\begin{aligned}
||(\pi_\omega(W(hf)) - \mathbb{1})\Phi_\omega(f)\Omega_\omega||^2 &= 2hb_2 - \sum_{k=3}^{\infty} b_k k h^{k-1} + 2(-h)b_2 - \sum_{k=3}^{\infty} b_k k(-h)^{k-1} \\
&= -\sum_{k=2}^{\infty} b_{k+1}(k+1)(h^k + (-h)^k) = -2\sum_{k=1}^{\infty} b_{2k+1}(2k+1)h^{2k} \ .
\end{aligned} \tag{5.108}$$

As before, we see that

$$\lim_{h\to 0}\left|\left|\frac{\pi_\omega(W(hf)) - \mathbb{1}}{h}\Phi_\omega(f)\Omega_\omega\right|\right|^2 = \sqrt{6|b_3|} < \infty \ . \tag{5.109}$$

The rest of the proof follows the same steps as the proof that $\Omega_\omega \in D(\Phi_\omega(f))$, and we conclude that, indeed, we have $\Omega_\omega \in D((\Phi_\omega(f))^2)$.

For general n, one wishes to find a bound for

$$\left|\left|\frac{\pi_\omega(W(hf)) - \mathbb{1}}{h}(\Phi_\omega(f))^{n-1}\Omega_\omega\right|\right| \ . \tag{5.110}$$

The steps of the proof are the same as the prior two cases, but, the third line in eq. (5.103) becomes significantly more cumbersome to deal with. One uses the multinomial theorem and the analytic states to conclude as before that a bound exists.

Thus far, we have have shown that if ω is an analytic state, then the corresponding cyclic vector Ω_ω of the regular cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ belongs to the class $C^\infty(\Phi_\omega(f))$ for any $f \in \mathcal{H}$. To conclude that Ω_ω truly is an analytic vector, we must show that

$$\sum_{k=0}^{\infty} \frac{t^n}{n!}||(\Phi_\omega(f))^n\Omega_\omega|| < \infty \ . \tag{5.111}$$

As we earlier stated, because the mapping $t \mapsto \omega(W(tf))$ is analytic, it is fully determined by the value of its derivatives at the origin. In particular, with reference to the prior proofs, we know that

$$\omega(W(tf)) = \sum_{k=0}^{\infty} \frac{t^n}{n!}\langle(i\Phi_\omega(f))^n\Omega_\omega, \Omega_\omega\rangle \ . \tag{5.112}$$

Selecting only the even terms, it is clear that the mapping

$$t \mapsto \sum_{k=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \left\langle (\Phi_\omega(f))^{2n} \Omega_\omega, \Omega \right\rangle \tag{5.113}$$

is analytic. This is easily seen by using the root test and the fact that the previous function was analytic. Because the operator $\Phi_\omega(f)$ is self-adjoint, we have

$$\left\langle (\Phi_\omega(f))^{2n} \Omega_\omega, \Omega \right\rangle = ||(\Phi_\omega(f))^n \Omega_\omega||^2 \ . \tag{5.114}$$

Again, using the root test, we see that the mapping

$$t \mapsto \sum_{k=0}^{\infty} \frac{t^{2n}}{(2n)!} ||(\Phi_\omega(f))^n \Omega_\omega||^2 \tag{5.115}$$

is analytic. Using Cauchy-Schawrz, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t^n}{n!} ||(\Phi_\omega(f))^n \Omega_\omega|| &= 1 + \sum_{k=1}^{\infty} \frac{1}{n} \frac{t^n}{(n-1)!} ||(\Phi_\omega(f))^n \Omega_\omega|| \\ &\leq 1 + \left(\frac{\pi^2}{6} \sum_{k=1}^{\infty} \frac{t^{2n}}{((n-1)!)^2} ||(\Phi_\omega(f))^n \Omega_\omega||^2 \right)^{\frac{1}{2}} \ . \end{aligned} \tag{5.116}$$

Using Stirling's formula, one has

$$\frac{(2n)!}{((n-1)!)^2} \leq \frac{e(2n)^{2n+\frac{1}{2}}}{e^{2n}} \frac{e^{2n-2}}{2\pi(n-1)^{2n-1}} = \frac{1}{\sqrt{2}\pi e} \frac{n^{2n+\frac{1}{2}}}{(n-1)^{2n-1}} 2^{2n} \ . \tag{5.117}$$

There exists N such that for $n \geq N$, we have

$$\frac{n^{2n+\frac{1}{2}}}{(n-1)^{2n-1}} \leq \sqrt{2}\pi e n^{\frac{3}{2}} \ . \tag{5.118}$$

The factor $\sqrt{2}\pi e$ is simply chosen for convenience, any constant greater than 1 would suffice. In particular, for $n \geq N$, we have

$$\frac{(2n)!}{((n-1)!)^2} \leq n^{\frac{3}{2}} 2^{2n} \ . \tag{5.119}$$

Now, consider the mapping

$$t \mapsto \sum_{n=1}^{\infty} \frac{n^{\frac{3}{2}} (2t)^{2n}}{(2n)!} ||(\Phi_\omega(f))^n \Omega_\omega||^2 \ . \tag{5.120}$$

Again, via the root test, and the analyticity of the previous mappings, this mapping is analytic. In particular, it is convergent. Using this, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{t^{2n}}{((n-1)!)^2} ||(\Phi_\omega(f))^n \Omega_\omega||^2 \\ &\leq \sum_{n=1}^{N} \frac{t^{2n}}{((n-1)!)^2} ||(\Phi_\omega(f))^n \Omega_\omega||^2 + \sum_{n=N+1}^{\infty} \frac{n^{\frac{3}{2}} (2t)^{2n}}{(2n)!} ||(\Phi_\omega(f))^n \Omega_\omega||^2 < \infty \ . \end{aligned} \tag{5.121}$$

We see that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} ||(\Phi_\omega(f))^n \Omega_\omega|| \leq 1 + \left(\frac{\pi^2}{6} \sum_{n=1}^{\infty} \frac{t^{2n}}{((n-1)!)^2} ||(\Phi_\omega(f))^n \Omega_\omega||^2 \right)^{\frac{1}{2}} < \infty \ . \tag{5.122}$$

We have finally shown the converse, if ω is an analytic state, then the cyclic vector Ω_ω is an analytic vector of $\Phi_\omega(f)$.

We remark that all analytic states in a neighbourhood containing the origin are actually analytic on a strip containing the real line. This is a consequence of the interplay between linear and multiplicative structure, one can liken it to the exponential function. Explicitly, let U be the neighbourhood containing the origin in which $t \mapsto \omega(W(tf))$ is analytic in. Let $z \in \mathbb{R} \cap U^C$. Let $w \in \mathbb{R}$ be such that $z - w \in U$. We have

$$\omega(W(zf)) = \langle \pi_\omega(W((z-w)f))\Omega_\omega, \pi_\omega(W(-wf))\Omega_\omega \rangle = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \langle (i\Phi_\omega(f))^n \Omega_\omega, W(-wf)\Omega_\omega \rangle \ . \tag{5.123}$$

Now, by unitarity of $W(\cdot)$, we simply derive the following bounds

$$\left| \left\langle (i\Phi_\omega(f))^{2n}\Omega_\omega, \pi_\omega(W(-wf))\Omega_\omega \right\rangle \right| \leq ||(\Phi_\omega(f))^n|| \tag{5.124}$$

and

$$\left| \left\langle (i\Phi_\omega(f))^{2n-1}\Omega_\omega, \pi_\omega(W(-wf))\Omega_\omega \right\rangle \right| \leq ||\Phi_\omega(f)\Omega_\omega|| ||(\Phi_\omega(f))^{n-1}\Omega_\omega|| \ . \tag{5.125}$$

Now, using the root test, again, on the term

$$a_n = \frac{\langle (i\Phi_\omega(f))^n \Omega_\omega, W(-wf)\Omega_\omega \rangle}{n!} \tag{5.126}$$

and recalling the fact that the mapping $t \mapsto \omega(W(tf))$ is analytic, it can be shown that $\omega(W(zf))$ is representable by a convergent power series, and thus, is analytic at this point.

5.3 Summary of the Regularity Conditions for States

The previous discussion shows that with stringent enough regularity conditions on a state ω on the CCR-algebra, one is able to extract an analytic framework in which to do physics in. To be more explicit, the corresponding Gelfand-Neimark-Segal(GNS) construction gives us a Hilbert space $\mathcal{H}_\omega$, a representation of the CCR-algebra π_ω onto a closed sub-algebra of $\mathcal{B}(\mathcal{H}_\omega)$, and a cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$. In physical terms, by simply specifying the Weyl relations in a C^*-algebra, we are able to generate a space in which there exists bounded operators satisfying the commutation relations, and, in addition, we have the existence of a vacuum vector.

To go further, once we have the analyticity of the state ω, we are also able to show that the state is fully determined by a family of expectations given by $\{\omega((\Phi_\omega(f))^n) : n \in \mathbb{N},\ f \in \mathcal{H}\}$, and the cyclic vector Ω_ω is actually an analytic vector of $\Phi_\omega(f)$ for any $f \in \mathcal{H}$.

Naturally, the operator $\Phi_\omega(f)$ can be written in terms of $a_\omega(f)$ and $a^*_\omega(f)$. Equivalently, the state is determined by the family of expectations $\{\omega((a_\omega(f))^n(a^*_\omega(f)))^m) : n, m \in \mathbb{N},\ f \in \mathcal{H}\}$.

6 Gibbs Grand Canonical Equilibrium State and the Formation of the Bose-Einstein Condensate

6.1 Elementary Conditions to Define the Gibbs State

To begin the discussion on the ideal Bose gas, we first specialize to a one particle Hilbert space $\mathcal{H}$ with a Hamiltonian H. The ideal Bose gas consists of non-interacting particles, and, as such, the natural setting of the problem is the symmetric Fock space $\mathcal{F}^{(+)}$ over $\mathcal{H}$ with a Hamiltonian given by the second quantization of H which will be denoted $d\Gamma(H)$.

The dynamics of this system for an initial state $\psi \in \mathcal{F}^{(+)}$ is given by the evolution

$$t \mapsto \Gamma(e^{-itH})\psi \ . \tag{6.1}$$

The notation $\Gamma(e^{-itH})$ refers to the second quantization of the unitary operator e^{-itH}. As we are approaching this problem from the operator algebraic perspective, the dynamics of the Weyl operators are of interest. The dynamics are given by the one parameter group of automorphisms

$$t \in \mathbb{R},\ A \in \mathcal{B}(\mathcal{F}^{(+)}) \mapsto \Gamma(e^{itH})A\Gamma(e^{-itH}) \ . \tag{6.2}$$

Let $\psi \in \mathcal{F}^{(+)}$. Recall that the finite particle vectors $F(\mathcal{H})$ are a dense set, and they are analytic vectors of $\Phi(\cdot)$. Let $\{\psi_n\}_{n\in\mathbb{N}} \in F(\mathcal{F}^{(+)})$ be a sequence such that $\psi_n \to \psi$. For any $f \in \mathcal{H}$, we have

$$W(e^{itH}f)\psi = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(i\Phi(e^{itH}f))^k}{k!}\psi_n \ . \tag{6.3}$$

By definition, we have

$$\Phi(e^{itH}f)\psi_n = \frac{a_+(e^{itH}f) + a_+^*(e^{itH}f)}{\sqrt{2}}\psi_n \ . \tag{6.4}$$

Next, let $f_1, ..., f_n \in \mathcal{H}$. By the definition of second quantization, we have

$$\begin{aligned}
\Gamma(e^{itH})a_+^*(f)(f_1 \otimes ... \otimes f_n) &= P_+(e^{itH})_{n+1}P_+^2 a^*(f)(f_1 \otimes ... \otimes f_n) \\
&= P_+(e^{itH})_{n+1}(n+1)^{\frac{1}{2}}(f \otimes f_1 \otimes ... \otimes f_n) \\
&= P_+(n+1)^{\frac{1}{2}}(e^{itH}f \otimes e^{itH}f_1 \otimes ... \otimes e^{itH}f_n) \\
&= P_+ a^*(e^{itH}f)(e^{itH})_n(f_1 \otimes ... \otimes f_n) \\
&= P_+ a^*(e^{itH}f)P_+^2(e^{itH})_n(f_1 \otimes ... \otimes f_n) \\
&= a_+^*(e^{itH}f)\Gamma(e^{itH})(f_1 \otimes ... \otimes f_n) \ .
\end{aligned}$$

This shows that for finite particle vectors ϕ, we have

$$\Gamma(e^{itH})a_+^*(f)\Gamma(e^{-itH})\phi = a_+^*(e^{itH}f)\phi \tag{6.5}$$

and, by taking the adjoint, we also have

$$\Gamma(e^{itH})a_+(f)\Gamma(e^{-itH})\phi = a_+(e^{itH}f)\phi \ . \tag{6.6}$$

Using the above, we have

$$\begin{aligned}
\Phi(e^{itH}f)\psi_n &= \Gamma(e^{itH}f)\left(\frac{a_+(f) + a_+^*(f)}{\sqrt{2}}\right)\Gamma(e^{-itH})\psi_n \\
&= \Gamma(e^{itH}f)\Phi(f)\Gamma(e^{-itH})\psi_n \ .
\end{aligned} \tag{6.7}$$

This shows that

$$\sum_{k=0}^{\infty} \frac{(i\Phi(e^{itH}f))^k}{k!}\psi_n = \Gamma(e^{itH})\sum_{k=0}^{\infty} \frac{(i\Phi(f))^k}{k!}\Gamma(e^{-itH})\psi_n \ . \tag{6.8}$$

We have skipped some relevant but trivial details in the above calculations. The first detail is that it is obvious that $\Gamma(e^{-itH})\psi_n \in F(\mathcal{H})$ from the definition of the second quantization, and thus $\Gamma(e^{-itH})\psi_n \in C^\infty(\Phi(\cdot))$. The second details is that because $\Gamma(e^{-itH})$ is unitary, we have

$$\sum_{k=0}^{\infty} \frac{1}{k!}||(\Phi(f))^k\Gamma(e^{-itH})\psi_n|| < \infty \ . \tag{6.9}$$

This follows from the exact same bounds we had in lemma 3.1. The conclusion of these observations is that

$$W(e^{itH}f)\psi = \lim_{n\to\infty} \Gamma(e^{itH})W(f)\Gamma(e^{-itH})\psi_n = \Gamma(e^{itH})W(f)\Gamma(e^{-itH})\psi \ . \tag{6.10}$$

Denote the one parameter group of $*$-automorphisms

$$t \in \mathbb{R},\ A \in \mathcal{B}(\mathcal{F}^{(+)}) \mapsto \Gamma(e^{itH})A\Gamma(e^{-itH}) \tag{6.11}$$

by $\tau_t(A)$. By the previous computation, we see that τ_t satisfies

$$\tau_t(W(f)) = W(e^{itH}f) \ . \tag{6.12}$$

This shows that the τ_t is a $*$-automorphism of the CCR-algebra. In fact, it is the unique $*$-automorphism with this property. A simple computation shows that the collection of operators $\{W(e^{itH}f) : f \in \mathcal{H}\}$ satisfy the Weyl relations. By the uniqueness theorem, there exists a unique isometric $*$-isomorphism from the C^*-algebra generated by $\{W(e^{itH}f) : f \in \mathcal{H}\}$ and the CCR-algebra on $\mathcal{H}$, this $*$-automorphism must thus be given by τ_t.

The formalism we have described here is usually called the "Heisenberg picture" of quantum mechanics. Given a state $\omega : \mathcal{B}(\mathcal{F}^{(+)}) \to \mathbb{C}$, the time evolution of the state is given by $\omega \circ \tau_t$. In a sense, we do not consider the time evolution of any individual element $\psi \in \mathcal{H}$, instead we are more interested in the general time evolution of states and operators.

Given $\mu \in \mathbb{R}$, we define the generalized Hamiltonian K_μ by

$$K_\mu = d\Gamma(H - \mu\mathbb{1}) \ . \tag{6.13}$$

This can also be written via the number operator as

$$K_\mu = d\Gamma(H) - \mu N \ . \tag{6.14}$$

Informally, the Gibbs grand canonical equilibrium state is defined on $\mathcal{B}(\mathcal{F}^{(+)})$ by

$$\omega(A) = \frac{\mathrm{Tr}(e^{-\beta K_\mu}A)}{\mathrm{Tr}(e^{-\beta K_\mu})} \ . \tag{6.15}$$

The following proposition gives us sufficient conditions for $e^{-\beta K_\mu}$ to be a trace class operator.

Proposition 6.1. *Let H be a self-adjoint operator on $\mathcal{H}$ and $\beta, \mu \in \mathbb{R}$.The operator $e^{-\beta K_\mu}$ is trace class on $\mathcal{F}^{(+)}$ if and only if the operator $e^{-\beta H}$ is trace class on $\mathcal{H}$ and $\beta(H - \mu\mathbb{1}) > 0$.*

Proof. The following proof is going to mainly utilize the abstract properties of Hilbert-Schmidt operators and the geometric series.

First, assume that $e^{-\beta H}$ is trace class and $\beta(H - \mu\mathbb{1}) \geq 0$. By the spectral theorem, the operator $e^{-\beta H}$ is positive, and, because it is trace class, it is also a compact operator. Self-adjoint compact operators are Hilbert-Schmidt operators by [12, p. 210]. By virtue of being a trace class self-adjoint operator, there exists an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a sequence of scalars $\{\lambda_n\}_{n\in\mathbb{N}}$ such that

$$e^{-\beta H}\phi_n = e^{-\beta\lambda_n}\phi_n \ , \tag{6.16}$$

and the eigenvalues satisfy

$$\lim_{n\to\infty} e^{-\beta\lambda_n} = 0 \ . \tag{6.17}$$

This implies that

$$\beta \lim_{n\to\infty} \lambda_n = \infty \ . \tag{6.18}$$

There are two possibilities either $\beta > 0$ and $\lim_{n\to\infty} \lambda_n = \infty$, or $\beta < 0$ and $\lim_{n\to\infty} \lambda_n = -\infty$. In both of these cases, the sequence of eigenvalues is either bounded above or below. For instance, if $\lim_{n\to\infty} \lambda_n = \infty$, then there exists N such that for $n \geq N$, we have

$$\lambda_n \geq 1 \ . \tag{6.19}$$

For all $m \in \mathbb{N}$, we have

$$\lambda_m \geq \min\{1, \min\{\lambda_k : k \in \{1, ..., N-1\}\}\} \ . \tag{6.20}$$

The proof is similar for the other case. Regardless, we have some boundedness condition. From here on out, we will assume that $\beta > 0$, the proof for the case $\beta < 0$ is the same.

Let $m \in \mathbb{N}$ and denote $\mathcal{H}_m^{(+)}$ to be the symmetric m-fold tensor product of $\mathcal{H}$. Using the direct sum structure of Fock space, We have the following simple observation

$$\operatorname{Tr}\left(e^{-\beta K_\mu}\right) = \sum_{m=0}^{\infty} \operatorname{Tr}_{\mathcal{H}_m^{(+)}}\left(e^{-\beta(H-\mu\mathbb{1})}\right) \ . \tag{6.21}$$

Next, we remark that the symmetric Fock space has a basis which can be given by operating on the vacuum vector with a suitable amount of creation operators with specifically chosen elements. To give some details, let I be some finite index, and define an equivalence relation $\sim$ on the I-fold tensor products $\otimes_{i=1}^I \psi_i$ such that $\otimes_{i=1}^I \psi_i \sim \otimes_{i=1}^I \phi_i$ if there exists a permutation $\pi \in S_I$ such that $\otimes_{i=1}^I \phi_i = \otimes_{i=1}^I \psi_{\pi(i)}$.

Let $\{\psi_k\}_{k\in\mathbb{N}}$ be an orthonormal sequence in the single particle space. We remark that the symmetric Fock space has a basis which consists of the representatives of the previously defined equivalence classes.

If we consider the orthonormal basis of the single particle space consisting of the eigenvectors of H, then we see that the equivalence classes can be fully determined by giving the energies and their multiplicities. Using this specific representation, one computes that

$$\operatorname{Tr}_{\mathcal{H}_m}^{(+)}\left(e^{-\beta(H-\mu\mathbb{1})}\right) = \sum_{\mathbf{n}\in\mathbb{N}_0^{\mathbb{N}}} e^{-\beta\left(\sum_{k=1}^{\infty}(\lambda_k-\mu)n_k\right)} \mathbb{1}\left(\sum_{k\in\mathbb{N}} n_k = m\right) \ . \tag{6.22}$$

continuing, we have

$$\operatorname{Tr}(e^{-\beta K_\mu}) = \sum_{\mathbf{n}\in\mathbb{N}_0^{\mathbb{N}}} e^{-\beta\left(\sum_{k=1}^{\infty}(\lambda_k-\mu)n_k\right)} \ . \tag{6.23}$$

Finally, one computes that

$$\sum_{\mathbf{n}\in\mathbb{N}_0^{\mathbb{N}}} e^{-\beta\left(\sum_{k=1}^{\infty}(\lambda_k-\mu)n_k\right)} = \prod_{m=1}^{\infty}\sum_{k=0}^{\infty} e^{-\beta(\lambda_m-\mu)k} = \prod_{m=1}^{\infty} \frac{1}{1-e^{-\beta(\lambda_m-\mu)}} \ . \tag{6.24}$$

To summarize, we have shown that

$$\operatorname{Tr}\left(e^{-\beta K_\mu}\right) = \prod_{m=1}^{\infty} \frac{1}{1-e^{-\beta(\lambda_m-\mu)}} \ . \tag{6.25}$$

We have

$$\prod_{m=1}^{\infty} \frac{1}{1 - e^{-\beta(\lambda_m - \mu)}} = e^{\sum_{m=1}^{\infty} \ln\left(1 - e^{-\beta(\lambda_m - \mu)}\right)} \tag{6.26}$$

Now, by the condition $\beta(H - \mu\mathbb{1}) > 0$, we must have $\lambda_m > \mu$. Earlier, we remarked that λ_m is necessarily bounded below, it follows that there exists $C > \mu$ such that $\lambda_m \geq C > \mu$. We thus have $0 \leq e^{-\beta(\lambda_m - \mu)} \leq e^{-\beta(C-\mu)} < 1$ and

$$\left| \ln(1 - e^{-\beta(\lambda_m - \mu)}) - \ln(1 - 0) \right| \leq \frac{e^{-\beta(\lambda_m - \mu)}}{1 - e^{-\beta(C-\mu)}} \ . \tag{6.27}$$

This implies that

$$e^{\sum_{m=1}^{\infty} \ln\left(1 - e^{-\beta(\lambda_m - \mu)}\right)} \leq e^{\frac{e^{\beta\mu}}{1 - e^{-\beta(C-\mu)}} \sum_{m=1}^{\infty} e^{-\beta\lambda_m}} = e^{\frac{e^{\beta\mu}}{1 - e^{-\beta(C-\mu)}} \mathrm{Tr}_{\mathcal{H}}(e^{-\beta H})} < \infty \ . \tag{6.28}$$

This implies that the infinite product converges, and we thus have

$$\mathrm{Tr}\left(e^{-\beta K_\mu}\right) = \prod_{m=1}^{\infty} \frac{1}{1 - e^{-\beta(\lambda_m - \mu)}} < \infty \ , \tag{6.29}$$

which implies that $e^{-\beta K_\mu}$ is trace class.

For the converse, assume that $e^{-\beta K_\mu}$ is trace class. We immediately have

$$\mathrm{Tr}_{\mathcal{H}}(e^{-\beta(H - \mu\mathbb{1})}) \leq \mathrm{Tr}\left(e^{-\beta K_\mu}\right) \ , \tag{6.30}$$

which implies that $e^{-\beta H}$ is trace class. Next, we compute the trace of $e^{-\beta K_\mu}$ slightly differently. We have

$$\mathrm{Tr}\left(e^{-\beta K_\mu}\right) = \sum_{m=0}^{\infty} e^{\beta\mu m} \, \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta H}) \ . \tag{6.31}$$

Now, because $e^{-\beta H}$ is trace class, we have an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ and a set of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ with the same properties as before. Suppose that there exists λ_k such that

$$\lambda_k \leq \mu \ . \tag{6.32}$$

Using the corresponding eigenvector $\otimes_{i=1}^{m} \phi_k$, and the trace of $e^{-\beta H}$, we have

$$e^{\beta m\mu} \, \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta H}) \geq e^{\beta m(\mu - \lambda_k)} \geq 1 \ . \tag{6.33}$$

Applying this to the formula for the trace of $e^{-\beta K_\mu}$, we have

$$\mathrm{Tr}\left(e^{-\beta K_\mu}\right) = \sum_{m=0}^{\infty} e^{\beta\mu m} \, \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta H}) \geq \sum_{m=0}^{\infty} 1 = \infty \ . \tag{6.34}$$

This is a contradiction, and we must have $\lambda_k > \mu$ for all λ_k. Let $\psi \in D(H)$ such that $||\psi|| = 1$. We have

$$\langle H\psi, \psi\rangle = \sum_{n=1}^{\infty} \lambda_n a_n \langle \phi_n, \psi\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_n a_n (a_m)^* \langle \phi_n, \phi_m\rangle = \sum_{n=1}^{\infty} \lambda_n |a_n|^2 > \mu \sum_{n=1}^{\infty} |a_n|^2 = \mu \ . \tag{6.35}$$

In the above $\{a_k\}_{k\in\mathbb{N}}$ are coefficients when we expand ψ in the basis ϕ_n. The above shows that

$$H - \mu > 0 \tag{6.36}$$

and since $\beta > 0$ was assumed, we have

$$\beta(H - \mu) > 0 \tag{6.37}$$

as desired. □

The previous proposition shows that we either choose $\beta > 0$ or $\beta < 0$, and, depending on this choice, μ is either smaller or larger than any eigenvalue of H. In particular, we see that μ cannot be a discrete eigenvalue of H.

From here on out, we consider only the case $\beta > 0$.

6.2 Extension of Gibbs State to Creation and Annihilation Operators

The current Gibbs equilibrium state we have defined is sorely lacking due to the fact that state is only defined on the C^*-algebra of bounded operators of $\mathcal{F}^{(+)}$. For instance, the current framework does not, seemingly, allow us to compute the expectation of the Hamiltonian because in most practical cases it is unbounded. To this cause, we will show that the Gibbs equilibrium state can be extended to include the creation and annihilation operators, and by doing so we open up a large class of possible unbounded operators which we can compute the expectation of Gibbs equilibrium state for.

From here on out, we will omit the $+$-sign from the operators $a_+(\cdot)$ and $a_+^*(\cdot)$. Instead, we will simply refer to them as $a(\cdot)$ and $a^*(\cdot)$.

Assume that one of the conditions in proposition 6.1 holds so that the Gibbs equilibrium state

$$\omega(A) = \frac{\mathrm{Tr}(e^{-\beta K_\mu} A)}{\mathrm{Tr}(e^{-\beta K_\mu})} \tag{6.38}$$

makes sense. Recall that for any $f \in \mathcal{H}$, and any $\psi \in \mathcal{H}_m^{(+)}$, we have the following estimate

$$||a(f)\psi|| \leq (m+1)^{\frac{1}{2}} ||f|| ||\psi|| \ . \tag{6.39}$$

Note that for $\psi \in \mathcal{H}_m^{(+)}$ we have $a(f)\psi \in \mathcal{H}_{m-1}^{(+)}$. For a collection of elements $\{f_i\}_{i=1}^n \in \mathcal{H}$, reiterating the prior estimate, we have

$$\left|\left|\prod_{i=1}^n a(f_i)\psi\right|\right| \leq \prod_{i=0}^{n-1} (m+1-i)^{\frac{1}{2}} \prod_{i=1}^n ||f_i|| ||\psi|| \leq \prod_{i=1}^n ||f_i|| (m+1)^{\frac{n}{2}} ||\psi|| \ . \tag{6.40}$$

Now, note that by the definition of the second quantization, for $\psi \in \mathcal{H}_m^{(+)}$, we have $e^{-\beta K_\mu}\psi \in \mathcal{H}_m^{(+)}$. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_m^{(+)}$. We have

$$\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\frac{\beta}{2}K_\mu} a^*(f_n)...a^*(f_1)a(f_1)...a(f_n)e^{-\frac{\beta}{2}K_\mu}) \qquad (6.41)\\
&= \sum_{k=1}^\infty \left\langle e^{-\frac{\beta}{2}K_\mu} a^*(f_n)...a^*(f_1)a(f_1)...a(f_n)e^{-\frac{\beta}{2}K_\mu}\psi_k, \psi_k \right\rangle \\
&= \sum_{k=1}^\infty \left|\left|\prod_{i=1}^n a(f_i)e^{-\frac{\beta}{2}K_\mu}\psi_k\right|\right|^2 \\
&\leq \prod_{i=1}^n ||f_i||^2 (m+1)^n \sum_{k=1}^\infty ||e^{-\frac{\beta}{2}K_\mu}\psi_k||^2 \\
&= \prod_{i=1}^n ||f_i||^2 (m+1)^n \sum_{k=1}^\infty \left\langle e^{-\beta K_\mu}\psi_k, \psi_k \right\rangle \\
&= \prod_{i=1}^n ||f_i||^2 (m+1)^n \, \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta K_\mu}) \ .
\end{aligned}$$

The previous computation is valid since the operator inside the trace is a positive and bounded operator on $\mathcal{H}_m^{(+)}$. In addition, the computation shows that it is also of trace-class. The computation relies on the fact that the trace of a positive and bounded operator is basis-independent regardless of whether the value is finite.

Define an operator $A_f : F(\mathcal{H}) \to \mathcal{F}^{(+)}$ by

$$A_f\psi = a(f_1)...a(f_n)e^{-\frac{\beta}{2}K_\mu}\psi \ . \tag{6.42}$$

The domain $F(\mathcal{H})$ is dense, and, as a result the adjoint of A_f is uniquely defined. We immediately see that $F(\mathcal{H}) \subset D(A_f^*)$ and thus $D(A_f^*)$ is dense. It follows that the operator A_f is closable.

We are going to show that A_f has a bounded closure. To begin with, we will derive an estimate concerning the operator A_f and the finite particle vectors. Let $\phi \in F(\mathcal{H})$ be arbitrary. By virtue of being a finite particle vector, the vector ϕ can be decomposed as

$$\phi = \sum_{i=1}^{j} a_i\phi_i \tag{6.43}$$

where $\phi_i \in \mathcal{H}_{m(i)}^{(+)}$ for some distinct $m(i) \in \mathbb{N}$. We have

$$||A_f\phi||^2 = \sum_{i=1}^{j} a_i \langle A_f\phi_i, A_f\phi\rangle = \sum_{l=1}^{j}\sum_{i=1}^{j} a_i(a_l)^* \left\langle A_f^*A_f\phi_i, \phi_l\right\rangle \ . \tag{6.44}$$

We remark that $A_f^*A_f\phi_i \in \mathcal{H}_{m(i)}^{(+)}$. Intuitively, this operator leaves the spaces $\mathcal{H}_m^{(+)}$ invariant. This is obvious since we create and annihilate the same amount of particles, and the second quantization leaves the subspaces invariant. This implies that

$$\sum_{l=1}^{j}\sum_{i=1}^{j} a_i(a_l)^* \left\langle A_f^*A_f\phi_i, \phi_l\right\rangle = \sum_{i=1}^{j} |a_i|^2 \left\langle A_f^*A_f\phi_i, \phi_i\right\rangle = \sum_{i=1}^{j} |a_i|^2||\phi_i||^2 \left\langle A_f^*A_f\frac{\phi_i}{||\phi_i||}, \frac{\phi_i}{||\phi_i||}\right\rangle \ . \tag{6.45}$$

Earlier, we derived the following estimate

$$\mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^*A_f) \leq \prod_{i=1}^{n} ||f_i||^2(m+1)^n \,\mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta K_\mu}) \ . \tag{6.46}$$

Again, recall that the trace of a bounded positive operator is independent of the chosen basis. For a separable Hilbert space and an element η such that $||\eta|| = 1$, there always exists an orthonormal basis which contains the element η. Applying this idea, we have

$$\left\langle A_f^*A_f\frac{\phi_i}{||\phi_i||}, \frac{\phi_i}{||\phi_i||}\right\rangle \leq \mathrm{Tr}_{\mathcal{H}_{m(i)}^{(+)}}(A_f^*A_f) \ . \tag{6.47}$$

The left hand side is a single term in the trace over an orthonormal basis containing the element $\frac{\phi_i}{||\phi_i||}$, and the inequality follows. We trivially have

$$\mathrm{Tr}_{\mathcal{H}_{m(i)}^{(+)}}(A_f^*A_f) \leq \sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^*A_f) \ . \tag{6.48}$$

Combining the previous two estimates, we have

$$\sum_{i=1}^{j} |a_i|^2||\phi_i||^2 \left\langle A_f^*A_f\frac{\phi_i}{||\phi_i||}, \frac{\phi_i}{||\phi_i||}\right\rangle \leq \sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^*A_f)\left(\sum_{i=1}^{j} |a_i|^2||\phi_i||^2\right) = ||\phi||^2 \sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^*A_f) \ . \tag{6.49}$$

To summarize, we have shown that

$$||A_f\phi|| \leq \left(\sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^* A_f)\right)^{\frac{1}{2}} ||\phi|| \ . \tag{6.50}$$

It remains to show that the sum on the right hand side converges. To this end, we apply eq. (6.41) to estimate

$$\sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^* A_f) \leq \prod_{i=1}^{n} ||f_i||^2 \sum_{m=0}^{\infty} (m+1)^n \, \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta K_\mu}) \ . \tag{6.51}$$

Recall that there exists $C > 0$ such that $H - \mu > C > 0$. We have

$$\mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta K_\mu}) = \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(e^{-\beta(H-\mu-C)} e^{-\beta C}) \leq e^{-\beta m C} \, \mathrm{Tr}(e^{-\beta(H-\mu-C)}) \ . \tag{6.52}$$

The derivatives of a geometric series are all finite, we thus have

$$\sum_{m=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_m^{(+)}}(A_f^* A_f) \leq \mathrm{Tr}(e^{-\beta(H-\mu-C)}) \prod_{i=1}^{n} ||f_i||^2 \sum_{m=0}^{\infty} (m+1)^n \left(e^{-\beta C}\right)^m < \infty \ . \tag{6.53}$$

Define

$$M := \left(\mathrm{Tr}(e^{-\beta(H-\mu-C)}) \sum_{m=0}^{\infty} (m+1)^n \left(e^{-\beta C}\right)^m\right)^{\frac{1}{2}} \ . \tag{6.54}$$

We have shown that for any $\phi \in F(\mathcal{H})$, we have

$$||A_f\phi|| \leq M \prod_{i=1}^{n} ||f_i|| ||\phi|| \ . \tag{6.55}$$

Now, we will show that the closure of A_f is bounded. Let $\psi \in D(\overline{A_f})$. It follows that there exists a sequence $\{\psi_k\}_{k\in\mathbb{N}} \in F(\mathcal{H})$ such that $\psi_k \to \psi$ and $A_f\psi_k \to \overline{A}_f\psi$. Applying the previous inequality, We have

$$||\overline{A_f}\psi|| = \lim_{k\to\infty} ||A_f\psi_k|| \leq M \prod_{i=1}^{n} ||f_i|| \lim_{k\to\infty} ||\psi_k|| = M \prod_{i=1}^{n} ||f_i|| ||\psi|| \ . \tag{6.56}$$

Since $F(\mathcal{H}) \subset D(A_f) \subset D(\overline{A_f})$, it follows that $\overline{A_f}$ is a bounded linear operator on a dense set, and, by the bounded linear extension theorem, it follows that we can uniquely extend $\overline{A_f}$ to be a bounded operator on the whole space $\mathcal{F}^{(+)}$.

We will now omit the line over $\overline{A_f}$ and instead we will refer to the operator $\overline{A}_f$ by A_f.

To summarize, we have shown that there exists a bounded operator A_f such that for the finite particle vectors $\psi \in F(\mathcal{H})$, we have

$$A_f\psi = a(f_1)...a(f_n)e^{-\frac{\beta}{2}K_\mu}\psi, \ B_f\psi = a^*(f_1)...a^*(f_n)e^{-\frac{\beta}{2}K_\mu}\psi \ . \tag{6.57}$$

Furthermore, we have the estimate

$$\left(\mathrm{Tr}(A_f^* A_f)\right)^{\frac{1}{2}} \leq \left(\mathrm{Tr}(e^{-\beta(H-\mu-C)}) \sum_{m=0}^{\infty} (m+1)^n \left(e^{-\beta C}\right)^m\right) \prod_{i=1}^{n} ||f_i|| := M_n \prod_{i=1}^{n} ||f_i|| \ . \tag{6.58}$$

Let $\{g_i\}_{i=1}^m \in \mathcal{H}$ be another set of elements and let A_g be the corresponding operator. One uses a Cauchy-Schwartz-like inequality to show that

$$|\omega(a^*(f_1)...a^*(f_n)a(g_m)...a(g_1))|^2 \leq \omega(a^*(f_1)...a^*(f_n)a(f_n)...a(f_1))\omega(a^*(g_1)...g^*(f_m)a(g_m)...a(g_1)) \ . \tag{6.59}$$

This inequality follows from the same computation as the Cauchy-Schwartz inequality for some hermitian form. There is an important property in the trace which we will now, and frequently in the future, use. For an eigenstate ψ of K_μ with eigenvalue $\lambda(\psi)$, and for a suitable operator T, we have

$$\begin{aligned} \left\langle e^{-\beta K_\mu}T\psi, \psi\right\rangle = \left\langle e^{-\frac{\beta}{2}K_\mu}T\psi, e^{-\frac{\beta}{2}K_\mu}\psi\right\rangle &= \left\langle e^{-\frac{\beta}{2}K_\mu}T\psi, e^{-\frac{\beta}{2}\lambda(\psi)}\psi\right\rangle \\ &= \left\langle e^{-\frac{\beta}{2}K_\mu}Te^{-\frac{\beta}{2}\lambda(\psi)}\psi, \psi\right\rangle \\ &= \left\langle e^{-\frac{\beta}{2}K_\mu}Te^{-\frac{\beta}{2}K_\mu}\psi, \psi\right\rangle \ . \end{aligned} \tag{6.60}$$

Applying the above, we have

$$\begin{aligned} \mathrm{Tr}(e^{-\beta K_\mu}a^*(f_1)...a^*(f_n)a(f_n)...a(f_1)) &= \mathrm{Tr}(e^{-\frac{\beta}{2}K_\mu}a^*(f_1)...a^*(f_n)a(f_n)...a(f_1)e^{-\frac{\beta}{2}K_\mu}) \\ &= \mathrm{Tr}(A_f^*A_f) < \infty \ . \end{aligned} \tag{6.61}$$

Along with the Cauchy-Schwartz-like inequality, we have

$$|\omega(a^*(f_1)...a^*(f_n)a(g_m)...a(g_1))| \leq M_n M_m \prod_{i=1}^n ||f_i|| \prod_{i=1}^m ||g_i|| \ . \tag{6.62}$$

The above property can be viewed as a continuity property of the state. We also remark that when we order the operators so that the creation operators are on the left, we refer to this as normal ordering.

This lengthy discussion shows that we can extend the Gibbs equilibrium state ω to include any monomials and polynomials of the creation and annihilation operators. However, one should pay attention to the amount of creation and annihilation operators. In particular, when tracing over the finite particle states, if there is a differing amount of annihilation and creation operators, the trace vanishes. This is of importance, since it this implies that we are only interested in states where we create and annihilate the same amount of particles.

6.3 Computation of Two-Point Correlations of the Extended Gibbs State

First, for any finite particle vector ψ, we will need the following algorithms. We have

$$e^{-\frac{\beta}{2}K_\mu}a^*(h)\psi = a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}h)e^{-\frac{\beta}{2}K_\mu}\psi \ , \tag{6.63}$$

and

$$a(h)e^{-\frac{\beta}{2}K_\mu}\psi = e^{-\frac{\beta}{2}K_\mu}a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}h)\psi \ . \tag{6.64}$$

The proofs of these algorithms follow from the same computation as eq. (6.5).

The two-point correlations are of particular importance. Let $f, g \in \mathcal{H}$. Using eq. (6.63), eq. (6.64), the commutation relations, eq. (6.60), and the cyclicity of the trace for trace-class operators, we have

$$\omega(a^*(f)a(g)) = \frac{\mathrm{Tr}(e^{-\beta K_\mu}a^*(f)a(g))}{\mathrm{Tr}(e^{-\beta K_\mu})} = \frac{\mathrm{Tr}(e^{-\frac{\beta}{2}K_\mu}a^*(f)a(g)e^{-\frac{\beta}{2}K_\mu})}{\mathrm{Tr}(e^{-\beta K_\mu})} \tag{6.65}$$

$$= \frac{\mathrm{Tr}(a(g)e^{-\frac{\beta}{2}K_\mu}e^{-\frac{\beta}{2}K_\mu}a^*(f))}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}(e^{-\frac{\beta}{2}K_\mu}a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g)a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f)e^{-\frac{\beta}{2}K_\mu})}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}(e^{-\frac{\beta}{2}K_\mu}a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g)e^{-\frac{\beta}{2}K_\mu})}{\mathrm{Tr}(e^{-\beta K_\mu})} + \left\langle g, e^{-\beta(H-\mu\mathbb{1})}f\right\rangle$$

$$= \frac{\mathrm{Tr}(e^{-\beta K_\mu}a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g))}{\mathrm{Tr}(e^{-\beta K_\mu})} + \left\langle g, e^{-\beta(H-\mu\mathbb{1})}f\right\rangle \ .$$

Iterating this procedure, for any $n \in \mathbb{N}$, we have

$$\omega(a^*(f)a(g)) = \sum_{k=1}^{n}\left\langle g, e^{-k\beta(H-\mu\mathbb{1})}f\right\rangle + \omega(a^*(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}g)) \tag{6.66}$$

$$= \left\langle g, \sum_{k=1}^{n} e^{-k\beta(H-\mu\mathbb{1})}f\right\rangle + \omega(a^*(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}g)) \ . \tag{6.67}$$

Using the spectral theorem, we have

$$\left\langle g, \sum_{k=1}^{n} e^{-k\beta(H-\mu\mathbb{1})}f\right\rangle = \int_{\sigma(H)} \langle g, dE(\lambda)f\rangle \sum_{k=1}^{n}\left(e^{-\beta(\lambda-\mu)}\right)^k \ . \tag{6.68}$$

This is an integral over a complex measure. Let $\langle g, dE(\lambda)f\rangle = \alpha(\lambda)d\beta(\lambda)$ be the polar decomposition of the complex measure. We remark that α is a function such that $|\alpha| = 1$ and β is a non-negative real measure. It follows that

$$\int_{\sigma(H)} \langle g, dE(\lambda)f\rangle \sum_{k=1}^{n}\left(e^{-\beta(\lambda-\mu)}\right)^k = \int_{\sigma(H)} d\beta(\lambda)\alpha(\lambda)\sum_{k=1}^{n}\left(e^{-\beta(\lambda-\mu)}\right)^k \ . \tag{6.69}$$

For all n, we have

$$\left|\alpha(\lambda)\sum_{k=1}^{n}\left(e^{-\beta(\lambda-\mu)}\right)^k\right| \le \sum_{k=1}^{\infty}\left(e^{-\beta(\lambda-\mu)}\right)^k = \frac{e^{-\beta(\lambda-\mu)}}{1-e^{-\beta(\lambda-\mu)}} \ . \tag{6.70}$$

Earlier, we showed that the strict inequality

$$e^{-\beta(\lambda-\mu)} \le C < 1 \tag{6.71}$$

holds. It follows that

$$\left|\alpha(\lambda)\sum_{k=1}^{n}\left(e^{-\beta(\lambda-\mu)}\right)^k\right| \le \frac{e^{-\beta(\lambda-\mu)}}{1-e^{-\beta(\lambda-\mu)}} \le \frac{1}{1-C} \ . \tag{6.72}$$

Recall that $\langle g, E(\sigma(H))f\rangle = \langle g, f\rangle$. The total variation of the measure β is thus

$$\beta(\sigma(H)) = \left|\int_{\sigma(H)} \langle g, dE(\lambda)f\rangle\,(\alpha(\lambda))^{-1}\right| \le \int_{\sigma(H)} |\langle g, dE(\lambda)f\rangle| \le ||f||||g|| \ . \tag{6.73}$$

It follows that

$$\int_{\sigma(H)} d\beta(\lambda) \frac{1}{1-C} \leq \frac{||f||||g||}{1-C} < \infty \tag{6.74}$$

and, by the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \int_{\sigma(H)} d\beta(\lambda)\alpha(\lambda) \sum_{k=1}^{n} \left(e^{-\beta(\lambda-\mu)}\right)^k &= \int_{\sigma(H)} d\beta(\lambda)\alpha(\lambda) \frac{e^{-\beta(\lambda-\mu)}}{1-e^{-\beta(\lambda-\mu)}} \\ &= \int_{\sigma(H)} \langle g, dE(\lambda) f\rangle \frac{e^{-\beta(\lambda-\mu)}}{1-e^{-\beta(\lambda-\mu)}} \\ &= \left\langle g, e^{-\beta(H-\mu\mathbb{1})} \left(1-e^{-\beta(H-\mu\mathbb{1})}\right)^{-1} f \right\rangle \ .\end{aligned} \tag{6.75}$$

Using eq. (6.62), we see that for all $n \in \mathbb{N}$, we have

$$\omega(a^*(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}g)) \leq M_1^2 \left|\left|e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}\right|\right|^2 ||f||||g|| \ . \tag{6.76}$$

Again, we have the strict inequality $H - \mu\mathbb{1} > 0$. By the spectral theorem, we have

$$\left|\left|e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}\right|\right| < 1 \ . \tag{6.77}$$

Because the bounded operators on a Hilbert space form a Banach algebra, we have

$$\left|\left|e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}\right|\right| \leq \left|\left|e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}\right|\right|^n \ . \tag{6.78}$$

It follows that

$$\lim_{n\to\infty} |\omega(a^*(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}f)a(e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}g))| \leq \lim_{n\to\infty} M_1^2 \left|\left|e^{-\frac{\beta}{2}n(H-\mu\mathbb{1})}\right|\right| ||f||||g|| = 0 \ . \tag{6.79}$$

Returning to eq. (6.66), we see that since the equation holds for all $n \in \mathbb{N}$, it also holds when $n \to \infty$, and, by the previous calculations, we have

$$\omega(a^*(f)a(g)) = \left\langle g, e^{-\beta(H-\mu\mathbb{1})} \left(1-e^{-\beta(H-\mu\mathbb{1})}\right)^{-1} f \right\rangle \ . \tag{6.80}$$

If we denote $z = e^{\beta\mu}$, we can rewrite the above as

$$\omega(a^*(f)a(g)) = \left\langle g, ze^{-\beta H} \left(1-ze^{-\beta H}\right)^{-1} f \right\rangle \ . \tag{6.81}$$

The parameter z is typically called the activity for physical reasons.

Next, we will show that the two points correlations are, in fact, the only relevant correlations. This implies that the grand canonical equilibrium state is a quasi-free state. Recall that the state vanishes if there is a differing amount of creation and annihilation operators. By largely the same argument as before, we compute

$$\omega\left(\prod_{i=1}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right) = \frac{\mathrm{Tr}\left(e^{-\beta K_\mu}\prod_{i=1}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right)}{\mathrm{Tr}(e^{-\beta K_\mu})} \tag{6.82}$$

$$= \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{2}K_\mu}\prod_{i=1}^n a^*(f_i)\prod_{i=1}^n a(g_i)e^{-\frac{\beta}{2}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(\prod_{i=1}^n a(g_i)e^{-\frac{\beta}{2}K_\mu}e^{-\frac{\beta}{2}K_\mu}\prod_{i=1}^n a^*(f_i)\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{2}K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=1}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)e^{-\frac{\beta}{2}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \sum_{j=1}^n \left\langle g_j, ze^{-\beta H}f_1\right\rangle \omega\left(\prod_{i=2}^n a^*(f_i)\prod_{i=1, i\neq j}^n a(g_i)\right)$$

$$+ \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{2}K_\mu}a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_1)\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)e^{-\frac{\beta}{2}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})} .$$

We continue the calculation of the second term

$$\frac{\mathrm{Tr}\left(e^{-\frac{\beta}{4}K_\mu}a^*(e^{-\frac{3\beta}{4}(H-\mu\mathbb{1})}f_1)e^{-\frac{\beta}{4}K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)e^{-\frac{\beta}{2}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{4}K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)e^{-\frac{\beta}{2}K_\mu}e^{-\frac{\beta}{4}K_\mu}a^*(e^{-\frac{3\beta}{4}(H-\mu\mathbb{1})}f_1)\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{4}K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)a^*(e^{-\frac{3\beta}{2}(H-\mu\mathbb{1})}f_1)e^{-\frac{3\beta}{4}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(e^{-\frac{\beta}{4}K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)a^*(e^{-\frac{3\beta}{2}(H-\mu\mathbb{1})}f_1)e^{-\frac{3\beta}{4}K_\mu}\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(\prod_{i=2}^n a^*(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}f_i)a^*(e^{-\frac{3\beta}{2}(H-\mu\mathbb{1})}f_1)e^{-\beta K_\mu}\prod_{i=1}^n a(e^{-\frac{\beta}{2}(H-\mu\mathbb{1})}g_i)\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \frac{\mathrm{Tr}\left(e^{-\beta K_\mu}a^*(e^{-\beta(H-\mu\mathbb{1})}f_1)\prod_{i=2}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right)}{\mathrm{Tr}(e^{-\beta K_\mu})}$$

$$= \omega\left(a^*(e^{-\beta(H-\mu\mathbb{1})}f_1)\prod_{i=2}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right) . \tag{6.83}$$

We thus have

$$\omega\left(\prod_{i=1}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right) = \sum_{j=1}^n \left\langle g_j, ze^{-\beta H}f_1\right\rangle \omega\left(\prod_{i=2}^n a^*(f_i)\prod_{i=1, i\neq j}^n a(g_i)\right) \tag{6.84}$$

$$+ \omega\left(a^*(e^{-\beta(H-\mu\mathbb{1})}f_1)\prod_{i=2}^n a^*(f_i)\prod_{i=1}^n a(g_i)\right) .$$

As before, we iterate this step k times to achieve

$$\omega\left(\prod_{i=1}^{n} a^*(f_i) \prod_{i=1}^{n} a(g_i)\right) = \sum_{j=1}^{n} \left\langle g_j, \sum_{m=1}^{k} z^m e^{-m\beta H} f_1 \right\rangle \omega\left(\prod_{i=2}^{n} a^*(f_i) \prod_{i=1, i\neq j}^{n} a(g_i)\right) \tag{6.85}$$
$$+ \omega\left(a^*(e^{-k\beta(H-\mu\mathbb{1})} f_1) \prod_{i=2}^{n} a^*(f_i) \prod_{i=1}^{n} a(g_i)\right) .$$

Letting $k \to \infty$, we have

$$\omega\left(\prod_{i=1}^{n} a^*(f_i) \prod_{i=1}^{n} a(g_i)\right) = \sum_{j=1}^{n} \langle g_j, ze^{-\beta H}(1 - ze^{-\beta H})^{-1} f_1 \rangle \, \omega\left(\prod_{i=2}^{n} a^*(f_i) \prod_{i=1, i\neq j}^{n} a(g_i)\right) \tag{6.86}$$
$$= \sum_{j=1}^{n} \omega(a^*(f_1) a(g_j)) \omega\left(\prod_{i=2}^{n} a^*(f_i) \prod_{i=1, i\neq j}^{n} a(g_i)\right) .$$

Using this recursion it is clear that the n-point correlations depend only on linear combinations of the two-point correlations. This implies that, to some extent, the Gibbs state is fully determined by its two point functions.

6.4 Value of the Weyl Operators for the Gibbs State and Summary of this Section

There is a slight problem with the Gibbs state in the form it is currently defined in. For future applications, one would like the Gibbs state to be an analytic state. We will momentarily focus on the mapping $t \mapsto \omega(W(tf))$. Computation of this mappings analyticity comes down to computing

$$\mathrm{Tr}(e^{-\beta K_\mu} W(tf)) = \sum_{N=0}^{\infty} \mathrm{Tr}_N(e^{-\beta K_\mu} W(tf)) . \tag{6.87}$$

The individual N-particle states are not problematic to deal with. Indeed, we compute

$$\mathrm{Tr}_N(e^{-\beta K_\mu} W(tf)) = \sum_{k\in\mathbb{N}} z^N e^{-\beta\lambda_k} \langle W(tf)\psi_k, \psi_k\rangle \tag{6.88}$$
$$= z^N \sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} \frac{(-1)^n e^{-\beta\lambda_k} t^{2n}}{(2n)!} ||\Phi(f)^n \psi_k||^2 .$$

We will need the estimate

$$\frac{||\Phi(f)^n \psi_k||^2}{(2n)!} \leq \frac{2^n (N+n)!}{(2n)! N!} ||f||^{2n} ||\psi_k||^2 . \tag{6.89}$$

It follows that

$$\sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} z^N \frac{e^{-\beta\lambda_k} |t|^{2n}}{(2n)!} ||\Phi(f)^n \psi_k||^2 \leq \mathrm{Tr}_N(e^{-\beta K_\mu}) \sum_{n\in\mathbb{N}} \frac{(2|t|^2)^n (N+n)!}{(2n)! N!} ||f||^{2n} .$$

Now, if we denote the above terms by x_n, then we have

$$\frac{x_{n+1}}{x_n} = \frac{2|t|^2 ||f||^2 (N+n+1)}{(2n+2)(2n+1)} . \tag{6.90}$$

This shows that

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \infty . \tag{6.91}$$

This implies that for all $t \in \mathbb{R}$, we have

$$\sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{N}} z^N \frac{e^{-\beta\lambda_k}|t|^{2n}}{(2n)!} \,||\Phi(f)^n\psi_k||^2 < \infty \ . \tag{6.92}$$

By Fubini's theorem, we have

$$\mathrm{Tr}(e^{-\beta K_\mu}W(tf)) = \sum_{N=0}^{\infty} \mathrm{Tr}_N(e^{-\beta K_\mu}W(tf)) = \sum_{N=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mathrm{Tr}_N(e^{-\beta K_\mu}\Phi(f)^{2n}) \ . \tag{6.93}$$

Using the same estimate as before, we have

$$\mathrm{Tr}_N(e^{-\beta K_\mu}\Phi(f)^{2n}) \le \frac{2^n(N+n)!\,\mathrm{Tr}_N(e^{-\beta K_\mu})||f||^{2n}}{N!} \ . \tag{6.94}$$

We wish to show the convergence of the series

$$\sum_{N=0}^{\infty}\sum_{n=0}^{\infty} \frac{|t|^{2n}}{(2n)!} \mathrm{Tr}_N(e^{-\beta K_\mu}\Phi(f)^{2n}) \le \sum_{N=0}^{\infty}\sum_{n=0}^{\infty} \frac{|t|^{2n}}{(2n)!} \frac{2^n(N+n)!\,\mathrm{Tr}_N(e^{-\beta K_\mu})||f||^{2n}}{N!} \ . \tag{6.95}$$

We can apply Fubini's theorem again, and integrating over the index N, we note the following

$$\begin{aligned} \frac{d^n}{dx^n}\sum_{N=0}^{\infty} x^N = \frac{d^n}{dx^n}\sum_{N=n}^{\infty} x^N &= \sum_{N=n}^{\infty} N(N-1)...(N-n+1)x^{N-n} \\ &= \sum_{N=0}^{\infty}(N+n)(N+n-1)...(N+1)x^N \\ &= \sum_{N=0}^{\infty} \frac{(N+n)!}{N!} x^N \ . \end{aligned} \tag{6.96}$$

Using the formula for the geometric series, we see that

$$\frac{n!}{(1-x)^n} = \sum_{N=0}^{\infty} \frac{(N+n)!}{N!} x^N \ . \tag{6.97}$$

It remains to see that

$$\sum_{N=0}^{\infty} \frac{(N+n)!\,\mathrm{Tr}(e^{-\beta K_\mu})}{N!} = \sum_{N=0}^{\infty} \frac{(N+n)!}{N!}(z\,\mathrm{Tr}(e^{-\beta H}))^N = \frac{n!}{(1-z\,\mathrm{Tr}(e^{-\beta H}))^n} \ . \tag{6.98}$$

Finally, again, using one of the various series convergence tests, one finds that

$$\sum_{n=0}^{\infty} \frac{|t|^{2n}}{(2n)!} \frac{2^n n!||f||^{2n}}{(1-z\,\mathrm{Tr}(e^{-\beta H}))^n} < \infty \tag{6.99}$$

for all $t \in \mathbb{R}$. It follows that we can apply Fubini's theorem to get

$$\omega(W(tf)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \omega(\Phi(f)^{2n}) \ . \tag{6.100}$$

In fact, because this equation holds for all $t \in \mathbb{R}$, we have actually shown that the mapping $t \mapsto \omega(W(tf))$ is an analytic mapping. In particular, we see that the Gibbs equilibrium state is an analytic state.

To finish this calculation, we will need some Wick's theorem-like tools. In particular, with reference to [16], we remark on the following structure of normally ordered operators and Hermitian polynomials. It is shown in the prior reference that if we have the following algebraic structure

$$[a, a^*] = \mathbb{1}, \; q = a + a^* \; , \tag{6.101}$$

then the normal ordering of the powers of q are given by

$$: q^n : = He_n(q) \; , \tag{6.102}$$

where He_n are the probabilists' Hermite polynomials.

As remarked in the text, the "traditional" way of defining the normal ordering is given by

$$: q^n : = \sum_{i=1}^{n} \binom{n}{i} (a^*)^{n-i} a^i \; . \tag{6.103}$$

Note that in our situation, when we act with ω on the normally ordered states, we will only be left with states which have the same amount of annihilation and creation operators. The "traditional" formula above already contains the combinatorial content that is relevant, that is, to say, that one needs to be careful while using the commutation relations, but, in the end, there will be a significant amount of cancellations among terms which leads to this simplification. This observation gives us the following

$$\omega(He_{2n}(q)) = \omega(: q^{2n} :) = \binom{2n}{n} \omega((a^*)^n a^n) \; . \tag{6.104}$$

Using the inverse formula for the Hermite polynomials, we have

$$q^{2n} = (2n)! \sum_{m=0}^{n} \frac{He_{2(n-m)}(q)}{2^m m!(2(n-m))} \; . \tag{6.105}$$

Combining the previous two relations, we have

$$\omega(q^{2n}) = (2n)! \sum_{m=0}^{n} \binom{2(n-m)}{n-m} \omega((a^*)^{n-m} a^{n-m}) \frac{1}{2^m m!(2(n-m))} = (2n)! \sum_{m=0}^{n} \frac{\omega((a^*)^{n-m} a^{n-m})}{2^m m!((n-m)!)^2} \; . \tag{6.106}$$

Now, we must be more precise with the operators a^* and a. We define

$$a^* := \frac{a^*(f)}{||f||}, \; a := \frac{a(f)}{||f||} \; . \tag{6.107}$$

It follows that

$$\omega((a^*)^{n-m} a^{n-m}) = \frac{(n-m)!}{||f||^{2(n-m)}} \omega(a^*(f)a(f))^{n-m} \; . \tag{6.108}$$

Plugging this in, we have

$$\omega(q^{2n}) = (2n)! \sum_{m=0}^{n} \frac{\omega(a^*(f)a(f))^{n-m}}{2^m ||f||^{2(n-m)} m!(n-m)!} \tag{6.109}$$

$$= \frac{(2n)!}{n!} \sum_{m=0}^{n} \left(\frac{1}{2}\right)^n \left(\frac{\omega(a^*(f)a(f))}{||f||^2}\right)^{n-m} \binom{n}{m} \tag{6.110}$$

$$= \frac{(2n)!}{n!} \left(\frac{1}{2} + \frac{\omega(a^*(f)a(f))}{||f||^2}\right)^n \; . \tag{6.111}$$

Now, we note that

$$\omega(q^{2n}) = 2^n ||f||^{2n} \omega(\Phi(f)^{2n}) \ . \tag{6.112}$$

It follows that

$$\omega(\Phi(f)^{2n}) = \frac{1}{2^n} \frac{(2n)!}{n!} \left(\frac{||f||^2}{2} + \omega(a^*(f)a(f)) \right)^n \ . \tag{6.113}$$

In the end, we have

$$\begin{aligned} \omega(W(tf)) &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \omega(\Phi(f)^{2n}) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \frac{1}{2^n} \frac{(2n)!}{n!} \left(\frac{||f||^2}{2} + \omega(a^*(f)a(f)) \right)^n \\ &= \exp\left(-\frac{t^2}{2} \left(\frac{||f||^2}{2} + \omega(a^*(f)a(f)) \right) \right) \\ &= \exp\left(-\frac{t^2}{4} \left\langle f, (1 + ze^{-\beta H})(1 - ze^{-\beta H})^{-1} f \right\rangle \right) \ . \end{aligned} \tag{6.114}$$

This entire discussion can be summarized in the following proposition.

Proposition 6.2. *Let H be a self-adjoint operator on $\mathcal{H}$ and assume that $e^{-\beta H}$ is trace class on $\mathcal{H}$ and $\beta(H - \mu \mathbb{1}) > 0$ for $\beta, \mu \in \mathbb{R}$.*

Define the Gibbs grand canonical equilibrium state $\omega : \mathcal{B}(\mathcal{F}^{(+)}) \to \mathbb{C}$ by

$$\omega(A) = \frac{\mathrm{Tr}(e^{-\beta K_\mu} A)}{\mathrm{Tr}(e^{-\beta K_\mu})} \ . \tag{6.115}$$

It follows that ω is the analytic state with two-point correlations given by

$$\omega(a^*(f)a(g)) = \left\langle g, ze^{-\beta H}(1 - ze^{-\beta H})^{-1} f \right\rangle \tag{6.116}$$

for any $f, g \in \mathcal{H}$, and, furthermore, the value of the state on the Weyl operator $W(f)$ is given by

$$\omega(W(f)) = e^{-\frac{\omega(\Phi(f)^2)}{2}} = e^{-\frac{\left\langle f, \left(1+ze^{-\beta H}\right)\left(1-ze^{-\beta H}\right)^{-1} f \right\rangle}{4}} \ . \tag{6.117}$$

Next, we will discuss the taking of the thermodynamic limit in multiple contexts.

6.5 A Theorem for Thermodynamic Limits with Fixed Activities

From here on out, we will consider the case $\mathcal{H} = L^2(\mathbb{R}^\nu)$, the square integrable functions on $\mathbb{R}^\nu$. Let H be the free particle Hamiltonian on $\mathcal{H}$. Let $\Lambda \subset \mathbb{R}^\nu$ be open and bounded. Denote H_Λ to be the free particle Hamiltonian restricted to the space Λ. We remark that the Hamiltonian H is already self-adjoint, whereas there are multiple self-adjoint extensions of H_Λ which correspond to different choices of boundary conditions.

For the following theorem, we will need a lemma from [11].

Lemma 6.1. *Let (H, σ) be a symplectic space and $\alpha : H \times H \to \mathbb{R}$ a symmetric positive bilinear form such that*

$$\sigma(f, g)^2 \leq \alpha(f, f)\alpha(g, g) \ . \tag{6.118}$$

Then there exists a state ϕ on the CCR-algebra of (H, σ) such that

$$\phi(W(f)) = e^{-\frac{1}{2}\alpha(f,f)} \ . \tag{6.119}$$

Proof. The proof can be found on [11, p. 22]. The proof is not involved and simply requires some properties of entrywise products of matrices. □

We have the following proposition.

Proposition 6.3. *Let H be the free particle Hamiltonian on the space $L^2(\mathbb{R}^\nu)$. For any open and bounded set $\Lambda \subset \mathbb{R}^\nu$, let H_Λ be one of the self-adjoint extensions of the restricted Hamiltonian corresponding to one of the classical boundary conditions. Let $\mathcal{A}_\Lambda$ be the CCR-algebra over $L^2(\Lambda)$.*

Let $\mathcal{A}$ be the CCR-algebra over the union of the spaces $L^2(\Lambda)$ where Λ goes over all of the open and bounded sets of $\mathbb{R}^\nu$.

Suppose that there exists $C \geq 0$ such that $H_\Lambda - \mu \geq C$ for all open and bounded $\Lambda \subset \mathbb{R}^\nu$. Let ω_Λ be the Gibbs grand canonical equilibrium state corresponding to the Hamiltonian H_Λ.

It follows that

$$\lim_{\Lambda' \to \infty} \omega_{\Lambda'}(A) = \omega(A) \ , \tag{6.120}$$

in the sense that Λ' will eventually contain any open and bounded set Λ, for all $A \in \mathcal{A}_\Lambda$ and all $\Lambda \subset \mathbb{R}^\nu$ open and bounded, where ω is the analytic state over $\mathcal{A}$ with the two point function given by

$$\omega(a_\omega^*(f)a_\omega(g)) = \left\langle g, ze^{-\beta H}(1 - ze^{-\beta H})^{-1} f \right\rangle = (2\pi)^{-\nu} \int_{\mathbb{R}^\nu} dp^\nu \overline{\widehat{g(p)}}\widehat{f(p)} ze^{-\beta p^2}(1 - ze^{-\beta p^2})^{-1} \ . \tag{6.121}$$

for any $f, g \in L^2(\mathbb{R}^\nu)$, and

$$\omega(W(f)) = e^{-\frac{\left\langle f, \left(1+ze^{-\beta H}\right)\left(1-ze^{-\beta H}\right)^{-1} f \right\rangle}{4}} \tag{6.122}$$

for any $f \in L^2(\mathbb{R}^\nu)$.

Proof. We will need to consider a new form of convergence for the following proof. In particular, we will utilize the concept of the strong graph limit and a theorem which relates strong graph convergence of generators to convergence of their semi-groups in an appropriate sense. For some definitions and background for strong graph limits, we suggest [12, p. 293].

First, recall that the infinitely differentiable functions with compact support $C_0^\infty(\mathbb{R}^\nu)$ are dense in $L^2(\mathbb{R}^\nu)$. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open and bounded sets. By increasing, we mean that for any open and bounded set Λ there will eventually be an n such that Λ will be contained in Λ_m for $m \geq n$. By definition, we immediately have

$$C_0^\infty(\mathbb{R}^\nu) \subset \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} D(H_{\Lambda_m}) \ . \tag{6.123}$$

Let $\psi \in C_0^\infty(\mathbb{R}^\nu)$. There exists $R > 0$ such that $\text{supp}(\psi) \subset B(0, R)$. Let m be large enough such $B(0, 2R) \subset \Lambda_m$. Recall that $H_{\Lambda_m} = H$ when restricted to $C_0^\infty(\Lambda_m)$. The support of ψ is contained in $B(0, R)$ which is contained in Λ_m. It follows that for large enough m, we have $||H\psi - H_{\Lambda_m}\psi|| = ||H\psi - H\psi|| = 0$. This implies that

$$H_{\Lambda_n}\psi \to H\psi \tag{6.124}$$

for all $\psi \in C_0^\infty(\mathbb{R}^\nu)$.

Next, let $\alpha > 0$ and define $G_\alpha = \lim_{n\to\infty} G(\mathbb{1} - \alpha(-iH_{\Lambda_n}))$. This notation means that G_α is the strong graph limit of $\mathbb{1} - \alpha(-iH_{\Lambda_n})$. The graph of G_α is defined to be the pairs $(\psi, \phi) \in L^2(\mathbb{R}^\nu) \times L^2(\mathbb{R}^\nu)$ such that there exists a sequence $\psi_n \in D(\mathbb{1} - \alpha(-iH_{\Lambda_n})) = D(H_{\Lambda_n})$ such that $\psi_n \to \psi$ and $\psi_n - \alpha(-iH_{\Lambda_n})\psi_n \to \phi$.

Let $\psi \in C_0^\infty(\mathbb{R}^\nu)$. By eq. (6.123), it follows that for large enough n, we have $\psi \in D(H_{\Lambda_n})$. We consider the constant sequence $\{\psi\}_{n\in\mathbb{N}}$. By eq. (6.124), we have

$$\psi - \alpha(-iH_{\Lambda_n})\psi \to \psi - \alpha(-iH)\psi \ . \tag{6.125}$$

We see that the for any $\psi \in C_0^\infty(\mathbb{R}^\nu)$, the pair $(\psi, \psi - \alpha(-iH)\psi)$ belongs to the graph of G_α. By definition, we immediately have $\psi - \alpha H\psi \in C_0^\infty(\mathbb{R}^\nu)$. In particular, this implies that $C_0^\infty \subset D(G_\alpha)$.

Next, let $\phi \in R(\mathbb{1} - \alpha(-iH))$. There exists $\psi \in D(H)$ such that $\phi = \psi - \alpha(-iH)\psi$. We remark that since H is self-adjoint, this range is dense and H is a closed operator with the special property that $C_0^\infty(\mathbb{R}^\nu)$ is a core for it. This implies that there exists a sequence of elements $\{\psi_n\}_{n\in\mathbb{N}} \in C_0^\infty(\mathbb{R}^\nu)$ such that $\psi_n \to \psi$ and $H\psi_n \to H\psi$.

Next, we remark that each element of the previous sequence has a compact support which is contained in some closed ball of finite radius. We construct a subsequence of this sequence by picking elements such that the supports of the sequence are monotonically increasing or decreasing. This is possible since every sequence of real numbers contains a monotonic subsequence. Here, we apply this idea to the supports. Denote this new sequence $\{\psi'_k\}_{k\in\mathbb{N}}$.

The only problem with this subsequence of our needs is that the supports might grow too fast. We define another subsequence $\{\psi''_k\}_{k\in\mathbb{N}}$. We define $\psi''_1 = \psi'_1$, for $n > 1$, if $\psi'_n \in C_0^\infty(\Lambda_n)$ then we set $\psi''_n := \psi'_n$, however if $\psi'_n \notin C_0^\infty(\Lambda_n)$ then we set $\psi''_n := \psi'_{n-1}$. For such a sequence, we have

$$\psi''_n \to \psi,\ \psi''_n - \alpha(-iH_{\Lambda_n}\psi''_n) = \psi''_n - \alpha(-iH\psi''_n) \to \psi - \alpha(-iH\psi) = \phi \ . \tag{6.126}$$

This implies that $\phi \in R(G_\alpha)$.

Since $C_0^\infty(\mathbb{R}^\nu) \subset D(G_\alpha)$ and $R(\mathbb{1} - \alpha(-iH)) \subset R(G_\alpha)$, it follows that both $D(G_\alpha)$ and $R(G_\alpha)$ are dense in $L^2(\mathbb{R}^\nu)$ and, by [2, p. 188, Theorem 3.1.28], it follows that there exists a self-adjoint operator S such that

$$e^{-iH_{\Lambda_n}t}\eta \to e^{-iSt}\eta \tag{6.127}$$

uniformly on finite intervals of t for all $\eta \in L^2(\mathbb{R}^\nu)$.

In addition to the previous property, the theorem also shows that the graph of G_α is, in fact, the graph of the operator $\mathbb{1} - \alpha S$. Earlier, we concluded that $(\psi, \psi - \alpha H\psi)$ belongs to the graph of G_α for $\psi \in C_0^\infty(\mathbb{R}^\nu)$. We see that $H = S$ when restricted to $C_0^\infty(\mathbb{R}^\nu)$. Both H and S are self-adjoint operators which agree on the core of H. Because H is the maximal symmetric extension, we must have $H = S$ and it follows that

$$e^{-iH_{\Lambda_n}t}\eta \to e^{-iHt}\eta \tag{6.128}$$

uniformly on finite intervals of t for all $\eta \in L^2(\mathbb{R}^\nu)$.

Now, let f be a bounded continuous function on $\mathbb{R}$. By [1, p. 52, Lemma 5.2.25], it follows that

$$f(H_{\Lambda_n})\psi \to f(H)\psi \tag{6.129}$$

for all $\psi \in L^2(\mathbb{R}^\nu)$. Define $f : [\mu + C, \mathbb{R}, \infty) \to \mathbb{R}$ by

$$f(x) := \frac{1 + e^{-\beta(x-\mu)}}{1 - e^{-\beta(x-\mu)}} \ . \tag{6.130}$$

We have

$$0 \leq f(x) \leq \frac{1+e^{-\beta C}}{1-e^{-\beta C}} = \frac{e^{\frac{\beta C}{2}}+e^{-\frac{\beta C}{2}}}{e^{\frac{\beta C}{2}}-e^{-\frac{\beta C}{2}}} = \coth\left(\frac{\beta C}{2}\right) < \infty \ . \tag{6.131}$$

It follows that f is a bounded function, and, as a result, we have

$$\left(1+ze^{-\beta H_{\Lambda_n}}\right)\left(1-ze^{-\beta H_{\Lambda_n}}\right)^{-1}\psi \to \left(1+ze^{-\beta H}\right)\left(1-ze^{-\beta H}\right)^{-1}\psi \tag{6.132}$$

for all $\psi \in L^2(\mathbb{R}^\nu)$.

Let $f \in L^2(\Lambda)$ for any Λ open and bounded, by the previous discussion, we have

$$\lim_{\Lambda'\to\infty}\omega_{\Lambda'}(W(f)) = \lim_{\Lambda'\to\infty} e^{-\frac{\left\langle f,\left(1+ze^{-\beta H_{\Lambda'}}\right)\left(1-ze^{-\beta H_{\Lambda'}}\right)^{-1}f\right\rangle}{4}} = e^{-\frac{\left\langle f,\left(1+ze^{-\beta H}\right)\left(1-ze^{-\beta H}\right)^{-1}f\right\rangle}{4}} = \omega(W(f)) \ . \tag{6.133}$$

The above shows that the the states converge pointwise for all generators of $\mathcal{A}$.

Using a similar argument, one also finds that the two point correlation functions converge to the desired state.

The fact that ω as defined by the pointwise limit of generators is a state follows from lemma 6.1. Indeed, define $\alpha : H \times H \to \mathbb{R}$ by

$$\alpha(f,g) := \frac{\left\langle f,\left(1+ze^{-\beta H}\right)\left(1-ze^{-\beta H}\right)^{-1}f\right\rangle}{2} \ . \tag{6.134}$$

Now, we have

$$\alpha(f,f) = \sum_{k\in\mathbb{N}} \frac{1+ze^{-\beta\lambda_k}}{1-ze^{-\beta\lambda_k}} |\langle\psi_k, f\rangle|^2 \ , \tag{6.135}$$

where ψ_k are eigenvectors of H and λ_k are the correspond eigenvalues.

Using the Cauchy-Shwartz inequality, we have

$$\left(\sum_{k\in\mathbb{N}} \frac{1+ze^{-\beta\lambda_k}}{1-ze^{-\beta\lambda_k}} |\langle\psi_k, f\rangle\langle\psi_k, g\rangle|\right)^2 \leq \alpha(f,f)\alpha(g,g) \ . \tag{6.136}$$

Next, note that

$$\frac{1+ze^{-\beta\lambda_k}}{1-ze^{-\beta\lambda_k}} \geq 1 \ , \tag{6.137}$$

and

$$\langle f,g\rangle = \sum_{k\in\mathbb{N}} \langle f,\psi_k\rangle\langle\psi_k,g\rangle \ . \tag{6.138}$$

It follows that

$$\sum_{k\in\mathbb{N}} \frac{1+ze^{-\beta\lambda_k}}{1-ze^{-\beta\lambda_k}} |\langle\psi_k, f\rangle\langle\psi_k, g\rangle| \geq \left|\sum_{k\in\mathbb{N}} \langle f,\psi_k\rangle\langle g,\psi_k\rangle\right| = |\langle f,g\rangle| \geq \operatorname{Im}\langle f,g\rangle = \sigma(f,g) \ . \tag{6.139}$$

It follows that

$$(\operatorname{Im}\langle f,g\rangle)^2 \leq \alpha(f,f)\alpha(g,g) \ , \tag{6.140}$$

and the conditions of lemma 6.1 are satisfied. It follows that ω as defined above is a state on the desired CCR-algebra. $\square$

6.6 Discussion on Bose-Einstein Condensation and Some Preliminary Lemmas

The previous theorem shows that one is able to take the thermodynamic limit in a rigorous and satisfactory manner. In particular, the state we obtain is the limit of finite volume states. There is, however, a fundamental problem in the assumptions of the previous theorem which are fundamental to the mathematical physics of formation of the Bose-Einstein condensate. The issue is with the assumption that $H_\Lambda - \mathbb{1}\mu \geq C\mathbb{1} > 0$.

Recall the definition of the number operator N. Let $\{f_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. The number operator acts on $\psi \in \mathcal{F}^{(+)}$ by

$$\langle N\psi, \psi\rangle = \sum_{n=1}^{\infty} \langle a^*(f_n)a(f_n)\psi, \psi\rangle \ . \tag{6.141}$$

Let $\{\psi_n\}_{n\in\mathbb{N}}$ be the eigenbasis of $\mathcal{H}$ corresponding to the operator H with eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$. We are going to now use the explicit form of the orthonormal basis that was mentioned in proposition 6.1. Let J be the collection of finite ordered subsequences of $\mathbb{N}$. Now, denote $I \subset J$ to be subsequences for which $C_{\mathbf{i}}a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega$ form an orthonormal basis of $\mathcal{F}^{(+)}$ where $\mathbf{i} \in I$ and $C_{\mathbf{i}}$ is a normalizing constant which depends on the eigenvectors and length of the subsequence. For such an orthonormal basis, we have

$$\begin{aligned}
\mathrm{Tr}(e^{-\beta K_\mu}N) &= \sum_{\mathbf{i}\in I} |C_{\mathbf{i}}|^2 \left\langle e^{-\beta K_\mu} N a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \sum_{\mathbf{i}\in I} |C_{\mathbf{i}}|^2 \left\langle N e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \sum_{\mathbf{i}\in I}\sum_{k=1}^{\infty} |C_{\mathbf{i}}|^2 ||a(f_k) e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega||^2 \ .
\end{aligned} \tag{6.142}$$

In the above, we made use of the spectral theorem to deduce that

$$\begin{aligned}
&\left\langle e^{-\beta K_\mu} N a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \left\langle N a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, e^{-\beta K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \left\langle N a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, e^{-\beta(\lambda_1+...+\lambda_{i_n}-\mu i_n)} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \left\langle N e^{-\frac{\beta}{2}(\lambda_1+...+\lambda_{i_n}-\mu i_n)} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, e^{-\frac{\beta}{2}(\lambda_1+...+\lambda_{i_n}-\mu i_n)} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \\
&= \left\langle N e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega, e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega\right\rangle \ .
\end{aligned} \tag{6.143}$$

By Fubini's theorem, we have

$$\begin{aligned}
\sum_{\mathbf{i}\in I}\sum_{k=1}^{\infty} |C_{\mathbf{i}}|^2 ||a(f_k) e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega||^2 &= \sum_{k=1}^{\infty}\sum_{\mathbf{i}\in I} |C_{\mathbf{i}}|^2 ||a(f_k) e^{-\frac{\beta}{2}K_\mu} a^*(\psi_{i_1})...a^*(\psi_{i_n})\Omega||^2 \\
&= \sum_{k=1}^{\infty} \mathrm{Tr}(e^{-\beta K_\mu} a^*(f_k)a(f_k)) \ .
\end{aligned} \tag{6.144}$$

It follows that

$$\omega(N) = \sum_{k=1}^{\infty} \omega(a^*(f_k)a(f_k)) \ , \tag{6.145}$$

for any orthonormal basis $\{f_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$.

Now, in the context of the previous proposition, let Λ be an open and bounded set of $\mathbb{R}^\nu$. We define the density

$$\rho_\Lambda(\beta, z) := \frac{\omega_\Lambda(N)}{|\Lambda|} \ . \tag{6.146}$$

Let $\{\psi_k\}_{k\in\mathbb{N}_0}$ be the eigenbasis corresponding to some classical self adjoint extension of the free particle Hamiltonian H_Λ on $L^2(\Lambda)$. Again, denote the corresponding eigenvalues by $\{\lambda_k\}_{k\in\mathbb{N}_0}$. By the previous proposition, we have

$$\rho_\Lambda(\beta, z) = \frac{1}{|\Lambda|}\sum_{k=0}^{\infty}\langle \psi_k, ze^{-\beta H_\Lambda}(1-ze^{-\beta H})^{-1}\psi_k\rangle = \frac{1}{|\Lambda|}\sum_{k=0}^{\infty} ze^{-\beta\lambda_k}(1-ze^{-\beta\lambda_k})^{-1} \ . \tag{6.147}$$

From the previous proposition, we can identify the volume of the state corresponding to the thermodynamic limit as

$$\rho(\beta, z) := (2\pi)^{-\nu}\int_{\mathbb{R}^\nu} dp^\nu ze^{-\beta p^2}(1-ze^{-\beta p^2})^{-1} \ . \tag{6.148}$$

This integral exists for $z \in [0, 1)$. By the monotone convergence theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^\nu} dp^\nu ze^{-\beta p^2}(1-ze^{-\beta p^2})^{-1} = \int_{\mathbb{R}^\nu} dp^\nu \sum_{k=1}^{\infty}(ze^{-\beta p^2})^k &= \sum_{k=1}^{\infty}\int_{\mathbb{R}^\nu} dp^\nu (ze^{-\beta p^2})^k \\ &= C(\nu)\sum_{k=1}^{\infty} z^k \int_0^\infty dr\ r^{\nu-1}e^{-\beta k r^2} \\ &= \frac{C(\nu)\Gamma\left(\frac{\nu}{2}\right)}{2\beta^{\frac{\nu}{2}}}\sum_{k=1}^{\infty}\frac{z^k}{k^{\frac{\nu}{2}}} \ ,\end{aligned} \tag{6.149}$$

where $C(\nu)$ is a constant appearing in the integration with spherical coordinates over $\mathbb{R}^\nu$.

From here on out, we will specialize to studying the self-adjoint extension of the bounded free particle Hamiltonian in a box of side length L with Dirichlet boundary conditions. To be explicit, we have $\Lambda_L =]-\frac{L}{2}, \frac{L}{2}[$, and the wave functions in the domain must vanish at the boundary of Λ_L. In this special case, the eigenvalues, or energies, are given by

$$\lambda_{\mathbf{k}} = \left(\frac{\pi}{L}\mathbf{k}\right)^2 \tag{6.150}$$

for $\mathbf{k} \in \mathbb{N}^\nu$.

We will now discuss one of the shortcomings of the assumption $H_{\Lambda_L} - \mu \geq C \geq 0$ for all $L > 0$. The eigenvalues of H_{Λ_L} are all strictly positive and bounded below by the smallest energy eigenvalue $\left(\frac{\pi}{L}\right)^2\nu$. Using the eigenvector decomposition, for any $\psi \in D(H_{\Lambda_L})$, we have

$$\langle H_{\Lambda_L}\psi, \psi\rangle = \sum_{\mathbf{k}\in\mathbb{N}^\nu}\lambda_{\mathbf{k}}|a_{\mathbf{k}}|^2||\psi_{\mathbf{k}}||^2 \geq \left(\frac{\pi}{L}\right)^2\nu||\psi|| \ . \tag{6.151}$$

Here $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^\nu}$ and $\{\psi_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^\nu}$ are the coefficients and orthonormal basis vectors formed from the eigenvectors of the Dirichlet Hamiltonian. This shows that $H_{\Lambda_L} \geq 0$ for all L, and we can choose $\mu_0 < 0$. It then follows that $H_{\Lambda_L} - \mu_0 \geq 0$ and for all $\mu < \mu_0$, we also have $H_{\Lambda_L} - \mu \geq 0$

In this case, we have $z = e^{\beta\mu} < 1$. By Cauchy-Schwartz we have

$$\left(\sum_{k=1}^{\infty}\frac{z^k}{k^{\frac{\nu}{2}}}\right)^2 \leq \sum_{k=1}^{\infty} z^{2k}\sum_{k=1}^{\infty}\frac{1}{k^\nu} = \frac{\zeta(\nu)z^2}{1-z^2} = \frac{\zeta(\nu)}{e^{-2\beta\mu}-1} \ , \tag{6.152}$$

where $\zeta(\cdot)$ is the Riemann zeta function. It follows that $\rho(\beta, z)$ is finite for $\nu \geq 2$. But, more importantly, if we study the parameter space $(\beta, \mu) \in (0, \infty) \times (-\infty, \mu_0]$, we see that the density $\rho(\beta, z)$ is a bounded function in this parameter space. Indeed, we have

$$\frac{1}{e^{-2\beta\mu} - 1} \leq \frac{1}{e^{-2\beta\mu_0} - 1} \ . \tag{6.153}$$

This implies that

$$\rho(\beta, z) \leq \frac{C(\nu)\Gamma\left(\frac{\nu}{2}\right)\zeta(\nu)}{2(2\pi)^\nu(e^{-2\beta\mu_0} - 1)} \ . \tag{6.154}$$

Let z be arbitrary again. If we examine the densities of the finite volumes, we immediately see that

$$\rho_{\Lambda_L}(\beta, z) \geq \frac{1}{L^\nu} \frac{ze^{-\beta\nu\left(\frac{\pi}{L}\right)^2}}{1 - ze^{-\beta\nu\left(\frac{\pi}{L}\right)^2}} \ . \tag{6.155}$$

Fix $L > 0$ and $\beta > 0$. If we choose μ such that z is close to $e^{\beta\nu\left(\frac{\pi}{L}\right)^2}$, then it is clear that $\rho_{\Lambda_L}(\beta, z)$ can be made arbitrarily large. There is thus a disconnect between the density of the state corresponding to the thermodynamic limit and density of a large but finite system.

This disconnect is fundamental to the formation of the Bose-Einstein condensate. We will show that there exists a critical density of the thermodynamic limit which clearly separates the gas into two distinct phases. Below the critical density, the thermodynamic limit behaves as we would expect, but at or above the critical density there is a new phase which occurs. This new phase is the Bose-Einstein condensate.

6.7 Convergence of Variable Activities at a Fixed Density

To begin with, we will consider the problem by examining what happens at a fixed density as we vary the activities of the finite volume states.

Let $\beta > 0$ and $0 \leq z \leq 1$. Note that in the case of $z = 1$, we must have $\nu \geq 3$, this is obvious by looking at eq. (6.149).

We will need the following simple observations of the monotonicity and convexity of the mapping $z \mapsto \rho_{\Lambda_L}(\beta, z)$.

By taking the first and second derivatives of the mappings $z \mapsto \rho(\beta, z)$ and $z \mapsto \rho_{\Lambda_L}(\beta, z)$, we see that both of these mappings are strictly increasing and convex.

Consider the following partitioning of $\mathbb{R}_+^\nu$ given by

$$\mathbb{R}_+^\nu = \bigcup_{\mathbf{k} \in \mathbb{N}^\nu} \prod_{i=1}^{n} \left] \frac{\pi(k_i - 1)}{L}, \frac{\pi k_i}{L} \right] \ . \tag{6.156}$$

If we denote

$$D_{\mathbf{k}} = \prod_{i=1}^{n} \left] \frac{\pi(k_i - 1)}{L}, \frac{\pi k_i}{L} \right] \tag{6.157}$$

then note that for $\mathbf{p} \in D_{\mathbf{k}}$, we have

$$\frac{ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}}{1 - ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}} \leq \frac{ze^{-\beta\mathbf{p}^2}}{1 - ze^{-\beta\mathbf{p}^2}} \ . \tag{6.158}$$

This can be written more suggestively as

$$\frac{ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}}{1-ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}} \leq \mathbb{1}(\mathbf{p}\in D_{\mathbf{k}})\frac{ze^{-\beta\mathbf{p}^2}}{1-ze^{-\beta\mathbf{p}^2}}\ . \tag{6.159}$$

Integrating over $D_{\mathbf{k}}$, we see that

$$\left(\frac{\pi}{L}\right)^{\nu}\frac{ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}}{1-ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}} \leq \int_{D_{\mathbf{k}}} dp^{\nu}\ \frac{ze^{-\beta\mathbf{p}^2}}{1-ze^{-\beta\mathbf{p}^2}}\ .$$

The sets $D_{\mathbf{k}}$ formed a partition of $\mathbb{R}_+^{\nu}$, it follows that

$$\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\left(\frac{\pi}{L}\right)^{\nu}\frac{ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}}{1-ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}} \leq \sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\int_{D_{\mathbf{k}}} dp^{\nu}\ \frac{ze^{-\beta\mathbf{p}^2}}{1-ze^{-\beta\mathbf{p}^2}} = \frac{1}{2^{\nu}}\int_{\mathbb{R}^{\nu}} dp^{\nu}\ \frac{ze^{-\beta\mathbf{p}^2}}{1-ze^{-\beta\mathbf{p}^2}}\ . \tag{6.160}$$

The last line follows from monotone convergence, and the ν-fold symmetry of changing the sign of any of the components of $\mathbf{p}$. It follows that

$$\rho_{\Lambda_L}(\beta,z) = \frac{1}{L^{\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}}{1-ze^{-\beta\left(\frac{\mathbf{k}\pi}{L}\right)^2}} \leq (2\pi)^{-\nu}\int_{\mathbb{R}^{\nu}} dp^{\nu}\ \frac{ze^{-\beta\mathbf{p}^2}}{1-ze^{-\beta\mathbf{p}^2}} = \rho(z,L)\ . \tag{6.161}$$

Using convexity, for $z_1, z_2 \in [0,1]$ such that $z_1 < z_2$, we have

$$\lim_{z\to z_1}\frac{\rho_{\Lambda_L}(\beta,z)-\rho_{\Lambda_L}(\beta,z_1)}{z-z_1} \leq \frac{\rho_{\Lambda_L}(\beta,z_2)-\rho_{\Lambda_L}(\beta,z_1)}{z_2-z_1} \leq \lim_{z\to z_2}\frac{\rho_{\Lambda_L}(\beta,z_2)-\rho_{\Lambda_L}(\beta,z)}{z_2-z}\ . \tag{6.162}$$

Computing the bounds, we have

$$\frac{1}{L^{\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{(e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}-z_1)^2} \leq \frac{\rho_{\Lambda_L}(\beta,z_2)-\rho_{\Lambda_L}(\beta,z_1)}{z_2-z_1} \leq \frac{1}{L^{\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{(e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}-z_2)^2}\ . \tag{6.163}$$

Setting $z_2 \in [0,1]$ and $z_1 = 0$, we have

$$\begin{aligned}\frac{\rho_{\Lambda_L}(\beta,z_2)}{z_2} \leq \frac{1}{L^{\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{(e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}-z_2)^2} &\leq \frac{1}{L^{\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{1}{e^{\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}\frac{1}{(1-z_2e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})^2} \\ &\leq \frac{1}{L^{\nu}}\frac{1}{1-z_2e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}\sum_{\mathbf{k}\in\mathbb{N}^{\nu}}\frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-z_2e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \\ &= \frac{\rho_{\Lambda_L}(\beta,z_2)}{z_2(1-z_2e^{-\beta\left(\frac{\pi}{L}\right)^2\nu})}\ .\end{aligned} \tag{6.164}$$

In the previous observations, we note that the eigenvalue corresponding to $\lambda_{(1,\dots,1)}$ is, naturally, the smallest energy eigenvalue. The same analysis as before can also be done for any self-adjoint extension for which the eigenvalues have a minimum. In light of this observation, we see that the remark concerning the arbitrarily large density of large but finite systems comes down to the fact that the μ can be made arbitrarily close to the smallest energy level. One might argue that in eq. (6.147) we could make μ close to, for instance, the second energy level, but, in that case, the condition $H_{\Lambda_L} - \mu \geq C \geq 0$ is not satisfied for any C, since the smallest eigenvalue will be smaller than μ, and, as a result, the previous proposition on taking the thermodynamic limit does not hold.

We summarize the previous observations into a lemma.

Lemma 6.2. *Let $\beta > 0$, and $z \in [0,1]$. For all $L > 0$, we have*

$$\rho_{\Lambda_L}(\beta, z) \leq \rho(\beta, z) \ . \tag{6.165}$$

We have

$$\lim_{L\to\infty} \rho_{\Lambda_L}(\beta), z) = \rho(\beta, z) \ . \tag{6.166}$$

The mappings $z \mapsto \rho_{\Lambda_L}(\beta, z)$ and $z \mapsto \rho(\beta, z)$ are strictly increasing and convex, and, as a result of the convexity, for any $z_1, z_2 \in [0,1]$ such that $z_1 < z_2$, we have

$$\frac{\partial \rho_{\Lambda_L}(\beta, z_1)}{\partial z_1} \leq \frac{\rho_{\Lambda_L}(\beta, z_2) - \rho_{\Lambda_L}(\beta, z_1)}{z_2 - z_1} \leq \frac{\partial \rho_{\Lambda_L}(\beta, z_2)}{\partial z_2} \tag{6.167}$$

and

$$\frac{\rho_{\Lambda_L}(\beta, z_1)}{z_1} \leq \frac{\partial \rho_{\Lambda_L}(\beta, z_2)}{\partial z_2} \leq \frac{\rho_{\Lambda_L}(\beta, z_2)}{z_2(1 - z_2 e^{-\beta\left(\frac{\pi}{L}\right)^2 \nu})} \ . \tag{6.168}$$

Proposition 6.4. *Let $\rho_{\Lambda_L}(\beta, z)$ be the density corresponding to the finite volume state with the free particle Hamiltonian restricted to a box with side length L with Dirichlet boundary conditions. Let $\rho(\beta, z)$ be the density of the thermodynamic limit of $\rho_{\Lambda_L}(\beta, z)$.*

Let $\overline{\rho} > 0$, and choose the unique z_L such that $\rho_{\Lambda_L}(\beta, z_L) = \overline{\rho}$. Define $\rho_c := \rho(\beta, 1)$.

1. *If $\overline{\rho} < \rho_c$ and $\overline{z}$ is the unique $\overline{z}$ such that $\rho(\beta, \overline{z}) = \overline{\rho}$, then*

$$\lim_{L\to\infty} z_L = \overline{z} \ . \tag{6.169}$$

2. *If $\overline{\rho} \geq \rho_c$ then*

$$\lim_{L\to\infty} z_L = 1 \ , \tag{6.170}$$

and

$$\lim_{L\to\infty} \frac{1}{L^\nu} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\right)^2 \nu}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\right)^2 \nu}} = \overline{\rho} - \rho_c \ . \tag{6.171}$$

Define

$$\rho_0 := \lim_{L\to\infty} \frac{1}{L^\nu} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\right)^2 \nu}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\right)^2 \nu}} \ . \tag{6.172}$$

The previous part can be restated as

$$\rho_0 = \overline{\rho} - \rho_c \ . \tag{6.173}$$

Proof. The following proof is tour of very basic analysis results and continuous usage of lemma 6.2.

First, we remark that the z_L and $\overline{z}$ are well-defined since the mappings $z \mapsto \rho_{\Lambda_L}(\beta, z)$ and $z \mapsto \rho(\beta, z)$ are strictly increasing by lemma 6.2. In the following proofs, we will frequently and liberally use lemma 6.2.

We will prove the first part of the proposition.

First, note that

$$\rho(\beta, \overline{z}) = \overline{\rho} = \rho_{\Lambda_L}(\beta, z_L) \leq \rho(\beta, z_L) \ . \tag{6.174}$$

It follows that $\overline{z} \leq z_L$. Now, we have

$$\frac{\rho_{\Lambda_L}(\beta, z_L) - \rho_{\Lambda_L}(\beta, \overline{z})}{z_L - \overline{z}} \geq \frac{\rho_{\Lambda_L}(\beta, \overline{z})}{\overline{z}} \ . \tag{6.175}$$

Rearranging, we have

$$\overline{\rho} - \rho_{\Lambda_L}(\beta, \overline{z}) \geq \frac{\rho_{\Lambda_L}(\beta, \overline{z})}{\overline{z}}(z_L - \overline{z}) \geq 0 \ . \tag{6.176}$$

Taking the limit as $L \to \infty$, we have

$$0 = \overline{\rho} - \rho(\beta, \overline{z}) \geq \frac{\rho(\beta, \overline{z})}{\overline{z}} \limsup_{L \to \infty}(z_L - \overline{z}) \geq 0 \ . \tag{6.177}$$

This implies that

$$\lim_{L \to \infty} z_L = \overline{z} \ , \tag{6.178}$$

as desired.

Next, we will prove the second part of he proposition.

We have

$$\overline{\rho} = \rho_{\Lambda_L}(\beta, z_L) \geq \rho_c = \rho(\beta, 1) \ . \tag{6.179}$$

It follows that $z_L \geq 1$. For each fixed z_L, we are taking the thermodynamic limit in accordance with proposition 6.3, as a result, we must have $\mu_L < \left(\frac{\pi}{L}\right)^2 \nu$. It follows that

$$z_L < e^{\beta \left(\frac{\pi}{L}\right)^2 \nu} \ , \tag{6.180}$$

from which we see that $\limsup_{L \to \infty} z_L \leq 1$, and, combining this with $z_L \geq 1$, we must have

$$\lim_{L \to \infty} z_L = 1 \ . \tag{6.181}$$

Next, define

$$\overline{\rho_0} = \limsup_{L \to \infty} \frac{1}{L^\nu} \frac{z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}}{1 - z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}} \tag{6.182}$$

and

$$\underline{\rho_0} = \liminf_{L \to \infty} \frac{1}{L^\nu} \frac{z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}}{1 - z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}} \ . \tag{6.183}$$

First, for any $\mathbf{m} \in \mathbb{N}^\nu$, we have the simple estimate

$$\begin{aligned} \frac{1}{L^\nu} \frac{z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}}{1 - z_L e^{-\beta \left(\frac{\pi}{L}\right)^2 \nu}} &\leq \rho_{\Lambda_L}(\beta, z_L) - \frac{1}{L^\nu} \sum_{\mathbf{k} \leq \mathbf{m}, \mathbf{k} \neq (1, \ldots, 1)} \frac{z_L e^{-\beta \left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta \left(\frac{\pi}{L}\mathbf{k}\right)^2}} \\ &\leq \overline{\rho} - \frac{1}{L^\nu} \sum_{\mathbf{k} \leq \mathbf{m}, \mathbf{k} \neq (1, \ldots, 1)} \frac{e^{-\beta \left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - e^{-\beta \left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \end{aligned} \tag{6.184}$$

In the previous computation, we made use of the fact that the mappings inside the sum are increasing functions in the variable z and the fact that $z_L \geq 1$. Next, we note that

$$\frac{1}{L^\nu} \sum_{\mathbf{k}\leq\mathbf{m},\mathbf{k}\neq(1,\ldots,1)} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} = \rho_{\Lambda_L}(\beta,1) - \frac{1}{L^\nu}\frac{e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} - \frac{1}{L^\nu}\sum_{\mathbf{k}>\mathbf{m}} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \tag{6.185}$$

We have

$$\frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \leq \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} \ . \tag{6.186}$$

We have

$$\sum_{\mathbf{k}\in\mathbb{N}^\nu} e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2} = \sum_{k_1=1}^{\infty}\cdots\sum_{k_\nu=1}^{\infty}\prod_{i=1}^{\nu} e^{-\beta\left(\frac{\pi}{L}\right)^2 k_i^2} \leq \left(\frac{1}{e^{\beta\left(\frac{\pi}{L}\right)^2}-1}\right)^\nu \ . \tag{6.187}$$

This computation shows that

$$\sum_{\mathbf{k}\in\mathbb{N}} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} < \infty \ , \tag{6.188}$$

and it follows that

$$\sum_{\mathbf{k}>\mathbf{m}} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \to 0,\ |\mathbf{m}|\to\infty \ . \tag{6.189}$$

We have shown that for any $\mathbf{m}$, we have

$$\frac{1}{L^\nu}\frac{z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} \leq \overline{\rho} - \rho_{\Lambda_L}(\beta,1) + \frac{1}{L^\nu}\frac{e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} + \frac{1}{L^\nu}\sum_{\mathbf{k}>\mathbf{m}} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \tag{6.190}$$

The above holds for any $\mathbf{m}\in\mathbb{N}^\nu$ and letting $m\to\infty$, we have

$$\frac{1}{L^\nu}\frac{z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} \leq \overline{\rho} - \rho_{\Lambda_L}(\beta,1) + \frac{1}{L^\nu}\frac{e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} \ . \tag{6.191}$$

Now, taking the limit superior, we have

$$\overline{\rho_0} \leq \overline{\rho} - \rho_c \ . \tag{6.192}$$

In the above, we used the fact that

$$\lim_{L\to\infty}\frac{1}{L^\nu}\frac{e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} = 0 \ , \tag{6.193}$$

for $\nu\geq 3$.

Next, as before, we write

$$\frac{1}{L^\nu}\frac{z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1-z_L e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} = \rho_{\Lambda_L}(\beta,z_L) - \frac{1}{L^\nu}\sum_{\mathbf{k}\in\mathbb{N}^\nu,\mathbf{k}\neq(1,\ldots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \tag{6.194}$$

For $\mathbf{k} \in \mathbb{N}^\nu, \mathbf{k} \neq (1, ..., 1)$, using eq. (6.180), we compute

$$\frac{1}{L^\nu} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \leq \frac{1}{L^\nu} \frac{e^{\beta\left(\frac{\pi}{L}\right)^2(\nu - \mathbf{k}^2)}}{1 - e^{\beta\left(\frac{\pi}{L}\right)^2(\nu - \mathbf{k}^2)}} \ . \tag{6.195}$$

For such $\mathbf{k}$, we naturally have $\mathbf{k}^2 > \nu$. It follows that

$$\lim_{L\to\infty} \frac{1}{L^\nu} \frac{e^{\beta\left(\frac{\pi}{L}\right)^2(\nu - \mathbf{k}^2)}}{1 - e^{\beta\left(\frac{\pi}{L}\right)^2(\nu - \mathbf{k}^2)}} = 0 \ . \tag{6.196}$$

Taking the limit inferior, we have

$$\underline{\rho_0} = \overline{\rho} - \limsup_{L\to\infty} \frac{1}{L^\nu} \sum_{\mathbf{k}\in\mathbb{N}^\nu, \mathbf{k}\neq(1,\dots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \tag{6.197}$$

Now, let $\mathbf{m} \in \mathbb{N}^\mu$ such that $\mathbf{m} \neq (1, ..., 1)$. We have

$$\sum_{\mathbf{k}\in\mathbb{N}^\nu, \mathbf{k}\neq(1,\dots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} = \sum_{\mathbf{k}<\mathbf{m}, \mathbf{k}\neq(1,\dots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} + \sum_{\mathbf{k}\geq\mathbf{m}, \mathbf{k}\neq(1,\dots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \tag{6.198}$$

Now, using eq. (6.196), the finite sum vanishes in the limit. Explicitly, we have

$$\begin{aligned} &\limsup_{L\to\infty} \frac{1}{L^\nu} \left(\sum_{\mathbf{k}<\mathbf{m}, \mathbf{k}\neq(1,\dots,1)} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} + \sum_{\mathbf{k}\geq\mathbf{m}} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \right) \\ &= \limsup_{L\to\infty} \frac{1}{L^\nu} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ . \end{aligned} \tag{6.199}$$

Combining with the earlier, we have

$$\underline{\rho_0} = \overline{\rho} - \limsup_{L\to\infty} \frac{1}{L^\nu} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \tag{6.200}$$

for any $\mathbf{m} \in \mathbb{N}^\nu$ such that $\mathbf{m} \neq (1, ..., 1)$.

Next, to save space, we define

$$\rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z) := \frac{1}{L^\nu} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \ , \tag{6.201}$$

and, as a result, we have

$$\underline{\rho_0} = \overline{\rho} - \limsup_{L\to\infty} \rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L) \ . \tag{6.202}$$

We remark that the mapping $z \mapsto \rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z)$ shares the same convexity and monotonicity properties in lemma 6.2. The proof of these facts are the same as in the lemma.

Using the convexity properties in lemma 6.2, we have

$$\rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L) - \rho^{(\mathbf{m})}_{\Lambda_L}(\beta, 1) \leq \frac{\rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L)(z_L - 1)}{z_L(1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2})} \ . \tag{6.203}$$

We remark that the $\mathbf{m}$ in the exponential appears because in the partial sum this is the smallest energy level. Rearranging, and using the inequality $\rho^{(\mathbf{m})}_{\Lambda_L}(\beta,1) \le \rho_{\Lambda_L}(\beta,1)$, for $\mathbf{m}^2 > 2\nu$, we have

$$\rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L) \le \frac{\rho^{(\mathbf{m})}_{\Lambda_L}(\beta,1) z_L \left(1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \le \frac{\rho_{\Lambda_L}(\beta,1) z_L \left(1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \ . \tag{6.204}$$

It remains to see that

$$\frac{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \le \frac{1 - e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}{1 - e^{2\beta\left(\frac{\pi}{L}\right)^2\nu} e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \ . \tag{6.205}$$

Taking the limit superior, we have

$$\limsup_{L\to\infty} \frac{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \le \lim_{L\to\infty} \frac{1 - e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}{1 - e^{2\beta\left(\frac{\pi}{L}\right)^2\nu} e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} = \frac{\beta(\pi\mathbf{m})^2}{\beta(\pi)^2(\mathbf{m}^2 - 2\nu)} = \frac{1}{1 - \frac{2\nu}{\mathbf{m}^2}} \ . \tag{6.206}$$

Combining this with the previous inequality, we have

$$\limsup_{L\to\infty} \rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L) \le \left(\lim_{L\to\infty} z_L \rho_{\Lambda_L}(\beta,1)\right) \left(\limsup_{L\to\infty} \frac{1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}\right) \le \rho_c \frac{1}{1 - \frac{2\nu}{\mathbf{m}^2}} \ . \tag{6.207}$$

Using this bound, for any $\mathbf{m} \in \mathbb{N}^\nu$, we have

$$\underline{\rho_0} = \overline{\rho} - \limsup_{L\to\infty} \rho^{(\mathbf{m})}_{\Lambda_L}(\beta, z_L) \ge \overline{\rho} - \rho_c \frac{1}{1 - \frac{2\nu}{\mathbf{m}^2}} \ . \tag{6.208}$$

Letting $\mathbf{m}^2 \to \infty$, we have

$$\underline{\rho_0} \ge \overline{\rho} - \rho_c \ . \tag{6.209}$$

Finally, we see that

$$\overline{\rho} - \rho_c \le \underline{\rho_0} = \liminf_{L\to\infty} \frac{1}{L^\nu} \frac{z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1 - z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} \le \limsup_{L\to\infty} \frac{1}{L^\nu} \frac{z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1 - z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} = \overline{\rho_0} \le \overline{\rho} - \rho_c \ . \tag{6.210}$$

The limit superior and inferior coincide. It follows that

$$\lim_{L\to\infty} \frac{1}{L^\nu} \frac{z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}}{1 - z e^{-\beta\left(\frac{\pi}{L}\right)^2\nu}} = \overline{\rho} - \rho_c \ , \tag{6.211}$$

as desired. □

6.8 Bose-Einstein Condensation as the Thermodynamic Limit of Finite-volume Systems with Varying Activities at a Fixed Density

In the previous proposition, we considered the density of the thermodynamic limit with variable activities. In particular, in the case where the density of the zeroth energy states is non-trivial, we noted that the activities converged to 1. The variable activities negate us from having a condition akin to $H_\Lambda - \mu \ge C > 0$ which stops us from being able to take the thermodynamic limit simply. This, however, is circumvented by a simple observation with a not-so-simple proof which we will omit. This property states that the function $L \mapsto \left\langle e^{-\beta H_{\Lambda_L}} f, f\right\rangle$ is increasing for any $f \in L^2(\Lambda')$ for Λ' open and bounded. For reference, we state this as a lemma.

Lemma 6.3. *For any $f \in L^2(\Lambda)$ for $\Lambda \subset \mathbb{R}^\nu$ open and bounded, the mapping*

$$L \mapsto \left\langle e^{-\beta H_{\Lambda_L}} f, f \right\rangle \tag{6.212}$$

is increasing.

Proof. The proof of this lemma involves the use of path integrals, and, as such, the proof will be omitted in order to keep the focus of this section on the formation of the condensate. The proof of this lemma can be found in [1, p. 376, Corollary 6.3.13]. □

We can now prove that the thermodynamic limit exists at the critical value $z = 1$.

Proposition 6.5. *For $\nu \geq 3$, the thermodynamic limit and correspond states in proposition 6.3 exist for the critical value $z = 1$.*

Proof. For $z = 1$ and $f \in \bigcup_{L>0} L^2(\Lambda_L)$, we have

$$\begin{aligned}\omega_{\Lambda_L}(W(f)) = e^{-\frac{1}{4}\left\langle f,(1+e^{-\beta H_{\Lambda_L}})(1-e^{-\beta H_{\Lambda_L}})^{-1} f\right\rangle} &= e^{-\frac{1}{4}\left\langle f,(1+e^{-\beta H_{\Lambda_L}})(1-e^{-\beta H_{\Lambda_L}})^{-1} f\right\rangle} \\ &= e^{-\frac{||f||^2}{4}} e^{-\frac{1}{4}\sum_{n=1}^{\infty}\left\langle f,e^{-n\beta H_{\Lambda_L}} f\right\rangle} .\end{aligned} \tag{6.213}$$

By the previous lemma, It follows that

$$\begin{aligned}\lim_{L\to\infty} \omega_{\Lambda_L}(W(f)) = e^{-\frac{||f||^2}{4}} e^{-\frac{1}{4}\sum_{n=1}^{\infty}\sup_{L>0}\left\langle f,e^{-n\beta H_{\Lambda_L}} f\right\rangle} &= e^{-\frac{||f||^2}{4}} e^{-\frac{1}{4}\sum_{n=1}^{\infty}\left\langle f,e^{-n\beta H} f\right\rangle} \\ &= e^{-\frac{1}{4}\left\langle f,(1+e^{-\beta H})(1-e^{-\beta H})^{-1} f\right\rangle} .\end{aligned} \tag{6.214}$$

The thermodynamic limit thus exists as before for the critical value $z = 1$. □

We can now specify and prove the ultimate theorem of this thesis. Namely, the existence of the two-phase system in which one of the phases is the Bose-Einstein condensate and the other phase is a regular non-interacting gas of Bosonic particles.

Proposition 6.6. *Let $\overline{\rho}, \beta > 0$. Let z_L be the unique root of $\rho_{\Lambda_L}(\beta, z_L) = \overline{\rho}$ and let $\overline{z}$ be the unique root of $\rho(\beta, \overline{z}) = \overline{\rho}$ if it exists. Furthermore, let ω_{Λ_L} be the Gibbs state corresponding to β and z_L.*

Define $\rho_c = \rho(\beta, 1)$.

Let $\mathcal{A}$ be the CCR-algebra over the union of the spaces $L^2(\Lambda_L)$. For any $A \in \mathcal{A}$, the weak-$*$ limit

$$\lim_{L\to\infty} \omega_{\Lambda_L}(A) = \omega(A) \tag{6.215}$$

exists, and the value of ω on the generators of $\mathcal{A}$ is given by

$$\omega(W(f)) = e^{-\frac{\rho_\omega(f,f)}{4}} \tag{6.216}$$

where, if $\overline{\rho} < \rho_c$, then

$$\rho_\omega(f, f) = \left\langle \frac{1 + \overline{z}e^{-\beta H}}{1 - \overline{z}e^{-\beta H}} f, f \right\rangle , \tag{6.217}$$

and, if $\overline{\rho} \geq \rho_c$, then

$$\rho_\omega(f, f) = 2^{\nu+1}\rho_0 \left| \int_{\mathbb{R}^\nu} d^\nu x f(x) \right|^2 + \left\langle \frac{1 + e^{-\beta H}}{1 - e^{-\beta H}} f, f \right\rangle , \tag{6.218}$$

where $\rho_0 = \overline{\rho} - \rho_c$.

Proof. Again the following proof is mostly a synthesis of basic analysis results and applications previous lemmas. The proof in itself is rather unenlightening, but the techniques used are surely worth studying.

In the following proofs, we will liberally use the properties of the mappings in lemma 6.2.

First, we consider the case $\overline{\rho} < \rho_c$. It follows that $\rho(\beta, \overline{z}) < \rho(\beta, 1)$, and we must have $\overline{z} < 1$. From proposition 6.4, we know that

$$\lim_{L\to\infty} z_L = \overline{z} < 1 \ . \tag{6.219}$$

It follows that for large enough L, we must have $z_L \leq \hat{z}$ and $\overline{z} \leq \hat{z}$ for some $\hat{z} < 1$.

Applying the Spectral theorem, we have

$$\left\langle \left(\frac{1+\overline{z}e^{-\beta H_{\Lambda_L}}}{1-\overline{z}e^{-\beta H_{\Lambda_L}}} - \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} \right) f, f \right\rangle = \left\langle \frac{2(\overline{z}-z_L)e^{-\beta H_{\Lambda_L}}}{(1-\overline{z}e^{-\beta H_{\Lambda_L}})(1-z_L e^{-\beta H_{\Lambda_L}})} f, f \right\rangle \ . \tag{6.220}$$

For $h > 0$, and L large enough, we have the trivial bounds

$$\left| \frac{2(\overline{z}-z_L)e^{-\beta h}}{(1-\overline{z}e^{-\beta h})(1-z_L e^{-\beta h})} \right| \leq \frac{2}{(1-\hat{z})^2} |\overline{z}-z_L| \ . \tag{6.221}$$

Combining the previous two statements, and using the Spectral theorem again, we have

$$\left| \left\langle \left(\frac{1+\overline{z}e^{-\beta H_{\Lambda_L}}}{1-\overline{z}e^{-\beta H_{\Lambda_L}}} - \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} \right) f, f \right\rangle \right| \leq \frac{2||f||^2}{(1-\hat{z})^2} |\overline{z}-z_L| \ . \tag{6.222}$$

To conclude, by proposition 6.4, in the limit as $L \to \infty$, we have

$$\begin{aligned} \lim_{L\to\infty} \left\langle \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} f, f \right\rangle = \lim_{L\to\infty} \left\langle \frac{1+\overline{z}e^{-\beta H_{\Lambda_L}}}{1-\overline{z}e^{-\beta H_{\Lambda_L}}} f, f \right\rangle &= \sup_{L>0} \left\langle \frac{1+\overline{z}e^{-\beta H_{\Lambda_L}}}{1-\overline{z}e^{-\beta H_{\Lambda_L}}} f, f \right\rangle \\ &= \left\langle \frac{1+\overline{z}e^{-\beta H}}{1-\overline{z}e^{-\beta H}} f, f \right\rangle \ . \end{aligned} \tag{6.223}$$

The last few lines follow from the fact that the inner product is an increasing function in the variable L, the reader can reference lemma 6.3 here. If we now consider, $\omega_{\Lambda_L}(W(f))$ from proposition 6.3, then we see that the previous statement implies that

$$\lim_{L\to\infty} \omega_{\Lambda_L}(W(f)) = \omega(W(f)) \tag{6.224}$$

where ω has the desired value on $W(f)$.

Next, we consider the case $\overline{\rho} \geq \rho_c$. We immediately have $\rho_{\Lambda_L}(\beta, 1) \leq \rho(\beta, 1) = \rho_c \leq \overline{\rho} = \rho_{\Lambda_L}(\beta, z_L)$. By monotonicity in the variable z, we must have $1 \leq z_L$. We begin by writing the inner product with the help of the eigenprojectors and eigenvalues given before. To be explicit, we have

$$\begin{aligned} \left\langle \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} f, f \right\rangle &= \sum_{\mathbf{k}\in\mathbf{N}^\nu} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}} \left| \int_{\mathbb{R}^\nu} d^\nu x \ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \\ &= \sum_{\mathbf{k}\in\mathbf{N}^\nu} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x \ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \ . \end{aligned} \tag{6.225}$$

First, by consideration of the explicit forms of the eigenvectors, it is well known that

$$\psi_{\mathbf{1},L}(x) = \left(\frac{2}{L}\right)^{\frac{\nu}{2}} \prod_{i=1}^{\nu} \cos\left(\frac{\pi x_i}{L}\right) \tag{6.226}$$

and

$$|\psi_{\mathbf{k},L}| \leq \left(\frac{2}{L}\right)^{\frac{\nu}{2}} \ . \tag{6.227}$$

We trivially have

$$L^{-\nu}\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 \leq 2^\nu\left(\int_{\mathbb{R}^\nu} d^\nu x\ |f(x)|\right)^2 \ . \tag{6.228}$$

The term on the right is finite because f is square integrable in an open and bounded set. We see that first term in the sum consists of two mappings of L with exiting limits. First, by proposition 6.4, we have

$$\lim_{L\to\infty}\frac{1+z_L e^{-\beta\nu\left(\frac{\pi}{L}\right)^2}}{L^\nu(1-z_L e^{-\beta\nu\left(\frac{\pi}{L}\right)^2})} = \overline{\rho}-\rho_c \ , \tag{6.229}$$

and, secondly, for all large enough L, we have

$$L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{1},L}(x)\right|^2 = 2^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\prod_{i=1}^{\nu}\cos\left(\frac{\pi x_i}{L}\right)\right|^2 \ . \tag{6.230}$$

By dominated convergence, we have

$$\lim_{L\to\infty} L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{1},L}(x)\right|^2 = 2^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\right|^2 \ . \tag{6.231}$$

It follows that if the limit of the inner product exists as $L\to\infty$, then we must have

$$\begin{aligned}\lim_{L\to\infty}\left\langle\frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}}f,f\right\rangle &= 2^\nu(\overline{\rho}-\rho_c)\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\right|^2 \\ &\quad + \lim_{L\to\infty}\sum_{\mathbf{k}\in\mathbf{N}^\nu,\ \mathbf{k}\neq\mathbf{1}}\frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})}L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 \ .\end{aligned} \tag{6.232}$$

The remainder of this proof will mirror the proof of the latter half of proposition 6.4 to derive the desired limit.

First, for $\mathbf{k}\neq\mathbf{1}$, one uses the same bounds as in eq. (6.180) and eq. (6.195) with eq. (6.228) to show that

$$\lim_{L\to\infty}\frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})}L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 = 0 \ . \tag{6.233}$$

To be completely explicit, the above limit as the product of two limits of which one is finite and the other is 0.

Now, to save space, we define

$$\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta,z) = \sum_{\mathbf{k}\geq\mathbf{m}}\frac{1+ze^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu(1-ze^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})}L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 \ . \tag{6.234}$$

By eq. (6.233), we have

$$\lim_{L\to\infty}\sum_{\mathbf{k}\in\mathbf{N}^\nu,\ \mathbf{k}\neq\mathbf{1}}\frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})}L^\nu\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 = \lim_{L\to\infty}\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta,z_L) \tag{6.235}$$

for any $\mathbf{m} \neq \mathbf{1}$. If the reader is uncomfortable with the use of the limit, even though it might not exist, the same result naturally holds for the superior and inferior limits.

Using monotonicity in the variable z, we have

$$\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) = \left\langle \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} f, f \right\rangle - \sum_{\mathbf{k}<\mathbf{m}} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \tag{6.236}$$

$$\geq \left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f \right\rangle - \sum_{\mathbf{k}<\mathbf{m}} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 .$$

Again, using eq. (6.233), we have

$$\liminf_{L\to\infty} \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) \geq \lim_{L\to\infty} \left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f \right\rangle = \sup_{L>0} \left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f \right\rangle = \left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f \right\rangle . \tag{6.237}$$

Here, we again used the monotonicity of the inner product in the variable L.

With the same techniques used in the proof of lemma 6.2, one can verify that the mapping $z \mapsto \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z)$ is convex, and we have

$$\begin{aligned}
\frac{\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) - \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, 1)}{z_L - 1} &\leq \frac{\partial \rho^{(\mathbf{m})}_{\omega,\Lambda_L}}{\partial z}(\beta, z_L) \qquad (6.238)\\
&= \sum_{\mathbf{k}\geq\mathbf{m}} \frac{e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)^2} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \\
&= \frac{1}{z_L} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)^2} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \\
&\quad - \frac{1}{z_L} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{1}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)^2} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \\
&\leq \frac{1}{z_L\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)} \sum_{\mathbf{k}\geq\mathbf{m}} \frac{1+z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}\right)} L^\nu \left| \int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x) \right|^2 \\
&= \frac{\rho^{(\mathbf{m})}_{\omega,\Lambda_L}}{z_L\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)} .
\end{aligned}$$

Continuing, we have

$$\begin{aligned}
\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) &\leq \frac{\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(z_L - 1)}{z_L\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)} + \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, 1) \qquad (6.239)\\
&\leq \frac{\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(z_L - 1)}{z_L\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)} + \left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f \right\rangle .
\end{aligned}$$

Rearranging, one finds that

$$\rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) \leq \frac{z_L\left(1-z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}\right)}{1-z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}} \left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f \right\rangle . \tag{6.240}$$

Now, we have already established an inequality for the limit superior of one of these equations on the right in eq. (6.206), indeed, for $\mathbf{m}^2 > 2\nu$, we have

$$\begin{aligned}\limsup_{L\to\infty} \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) &\leq \left(\limsup_{L\to\infty} \frac{z_L(1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2})}{1 - z_L^2 e^{-\beta\left(\frac{\pi}{L}\mathbf{m}\right)^2}}\right) \lim_{L\to\infty}\left(\left\langle \frac{1+e^{-\beta H_{\Lambda_L}}}{1-e^{-\beta H_{\Lambda_L}}} f, f\right\rangle\right) \\ &\leq \frac{1}{1-\frac{2\nu}{\mathbf{m}^2}}\left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f\right\rangle \ .\end{aligned} \tag{6.241}$$

Combining all of these inequalities, we have, for any $\mathbf{m} \in \mathbb{N}^\nu$, the inequalities

$$\left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f\right\rangle \leq \liminf_{L\to\infty} \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) \leq \limsup_{L\to\infty} \rho^{(\mathbf{m})}_{\omega,\Lambda_L}(\beta, z_L) \leq \frac{1}{1-\frac{2\nu}{\mathbf{m}^2}}\left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f\right\rangle \ . \tag{6.242}$$

Applying these bounds to eq. (6.232), and letting $\mathbf{m} \to \infty$, we have

$$\lim_{L\to\infty} \sum_{\mathbf{k}\in\mathbf{N}^\nu,\ \mathbf{k}\neq\mathbf{1}} \frac{1 + z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2}}{L^\nu(1 - z_L e^{-\beta\left(\frac{\pi}{L}\mathbf{k}\right)^2})} L^\nu \left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\psi_{\mathbf{k},L}(x)\right|^2 = \left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f\right\rangle \ . \tag{6.243}$$

We have thus shown that

$$\lim_{L\to\infty} \left\langle \frac{1+z_L e^{-\beta H_{\Lambda_L}}}{1-z_L e^{-\beta H_{\Lambda_L}}} f, f\right\rangle = 2^\nu(\overline{\rho} - \rho_c)\left|\int_{\mathbb{R}^\nu} d^\nu x\ f(x)\right|^2 + \left\langle \frac{1+e^{-\beta H}}{1-e^{-\beta H}} f, f\right\rangle \ . \tag{6.244}$$

Finally, if we now consider the form of ω_{Λ_L} from proposition 6.3, we have shown that

$$\lim_{L\to\infty} \omega_{\Lambda_L}(W(f)) = \omega(W(f)) \ , \tag{6.245}$$

and ω has the desired value for $W(f)$. □

7 Conclusion and Further Discussion

7.1 Bose-Einstein Condensation

The ultimate result of the previous section proposition 6.6 states that with a very specific process of taking the thermodynamic limit, namely, we must vary the activity to account for a change in the volume so as to maintain a fixed density, one is able to construct a state which presents two distinct phases of matter depending on the fixed density.

The first state, which occurs if we are under the critical density, is clearly just a regular non-interacting gas. The second state is the condensate. By comparison of the ρ_ω function in proposition 6.6, we see that above the critical density there is the addition of a scalar term. The effect of this additional scalar term is that physically one tries to add additional particles to the system, which corresponds to increasing the fixed density $\overline{\rho}$, instead of distributing "evenly", in some sense, they will immediately go to the lowest energy state and increase this scalar term.

Mathematically, we have shown that by using Gibbs grand canonical equilibrium states, and an appropriate limiting procedure, there exists a well-defined state on a physically relevant algebra of operators which shows a distinct splitting of the regular gas and condensate phases.

7.2 Mathematical Content of this Thesis

The first half of the thesis was used to construct and show that there is a rather broad and rich algebraic theory surrounding the field of quantum statistical mechanics. In a sense, one can consider the first half of this thesis to be a light introduction to certain field-theoretic arguments and methods.

The main result of the first half of the algebraic part of this thesis is theorem 4.2. By virtue of being a uniqueness theorem, the theorem roughly states that for certain algebraic and topological properties, it is enough to study any C^*-algebra which is generated by the Weyl operators. Going further, if one wishes to extract abstract versions of the annihilation and creation operators, with the help of additional regularity conditions in the form of analyticity and strong continuity of the represented forms of the time evolutions given by the Weyl operators, then we are able to do so as shown in section 5.2.

One of the interesting facts in the first half of the thesis concerned the analytic states in section 5.2. One will recall that the analytic states were regular states on some CCR-algebra which satisfied additional regularity properties in the form of analyticity of the state acting on a generator. Remarkably, these conditions were sufficient to yield a Hilbert space, a representation and an analytic vacuum vector of the state.

One can specify the commutation relations in the form of Weyl operators, then consider a state on the algebra which has some physical significance, and hope that this state is analytic. In this way one avoids talking about the concrete Hilbert space and vacuum vector, and, they are instead given for free. The issue here is that the construction of the Hilbert space and vacuum vector are a part of the GNS construction. In practice, one could try and explicitly compute what the GNS construction yields and identify the Hilbert space and vacuum vector with something concrete. Of course, in this case, one would presumably also already be able to approach the problem without the GNS construction.

The latter half of the thesis could actually be considered an example of this sort of approach. In the latter half, we did an explicit construction and extension of the Gibbs equilibrium state to include the annihilation and creation operators. In doing so, through various computations, we were able to calculate the value of the Weyl operator for rather general Hilbert space. To specialize to taking the thermodynamic limit, we of course had to include such concepts as volume which show up in the free particle Hamiltonian in a open and bounded set. Ultimately, we constructed the state corresponding to the two-phase gas, in which one of the phases was the condensate.

Having constructed these states, one might be tempted to try and generalize the abstract properties of these states rather than trying to come up with a new Hilbert space and Hamiltonian to study. We will discuss this idea in the next subsection.

The previous paragraphs give a succinct, but shallow, summary of the second half of the thesis. One notes that the first mathematical issues that one faces is the extension of the Gibbs state to the polynomials of annihilation and creation operators. This problem is dealt with by showing that $e^{-\beta K_\mu}$, along with some additional properties, is sufficiently "small", in some operator norm sense, so as to counteract the unboundedness of the creation and annihilation operators. This informal statement is made rigorous in section 6.2.

Next, one must contend with doing calculations inside the trace without the help of bounded operators. To this end, we make frequent applications of the method of computation used in eq. (6.60). For an example of the usage of this method, we suggest filling in some of the details of eq. (6.82). Finally, one computes the value of the Weyl operators using methods from [16], and shows convergence of states via various technical lemmas.

We remark that the mathematics used in the computations starting from section 6.5, barring some of the more technical results such as lemma 6.3 or the strong-graph limit theorem used in proposition 6.3, is relatively standard and basic. For the most part, we are simply computing upper and lower bounds, and using basic analysis.

7.3 Further Work

The ultimate constructed state in proposition 6.6 of this thesis shows a clear and physical splitting of phases. A natural point from where to generalize is to consider states in which there is more than a single scalar addition to the density. For instance, in this example of the condensate, there is a single energy level to which the particles all fall to. But what is stopping us from having a finite convex combination of other possible "ground states" in the condensate? In fact, it is trivial to see that one can write an analytic state which implements this precise idea.

It is less trivial to see how one would arrive at this state without just explicitly giving the value of the Weyl operators. In a sense, we would like to write something of a generalization to the single species condensate in which there is only a single ground state to which the particles flow to.

Another direction one could take is to consider different Hamiltonians. For instance, one can consider the ultrarelativistic gas in which the free particle Hamiltonian is given by $H_{\Lambda_L} = p_{\Lambda_L}$. There are certain relevant theorems such as the theorem concerning strong graph limit in proposition 6.3 which initially seem to be applicable to this case as well. The biggest possible problem are the non-trivial theorems concerning the path integrals in lemma 6.3. Nevertheless, one can take any relevant Hamiltonian and use the ideas of the thermodynamic limit and some of the computations as a guide to how to approach this problem with another Hamiltonian.

The mathematical possibilities with the algebraic approach are endless. One can casually peruse the first section of [1] and find numerous interesting and relevant topics to study and clarify.

To be more specific, for readers interested in studying the algebraic side of quantum statistical mechanics, we suggest first gaining an understanding of W^*-algebras and semi-group theory from [2]. One can then study the algebraic results which were left out from this thesis in [1, 5.2. Continuous Quantum Systems I]. A natural continuation of this thesis would be the next section of [1] which focuses on KMS-states. In fact, this thesis used a form of the KMS-condition given in eq. (6.5). However, the approach used in [1, 5.3. KMS States] is considerable more abstract and begins with C^*-dynamical systems.

For readers interested in studying the physics of quantum statistical mechanics, we recommend the classic text [8]. For something related to quantum statistical mechanics, one could study the quantum Bose liquid and its relation to superfluidity in [8][Part 2, Section 3]. For a modern, and thoroughly advanced, book on the kinetic theory of Bose-condensed gases, we suggest [5].

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